**EE 102** # Homework #8 Solutions

**Fall 2001** **Professor Paganini**

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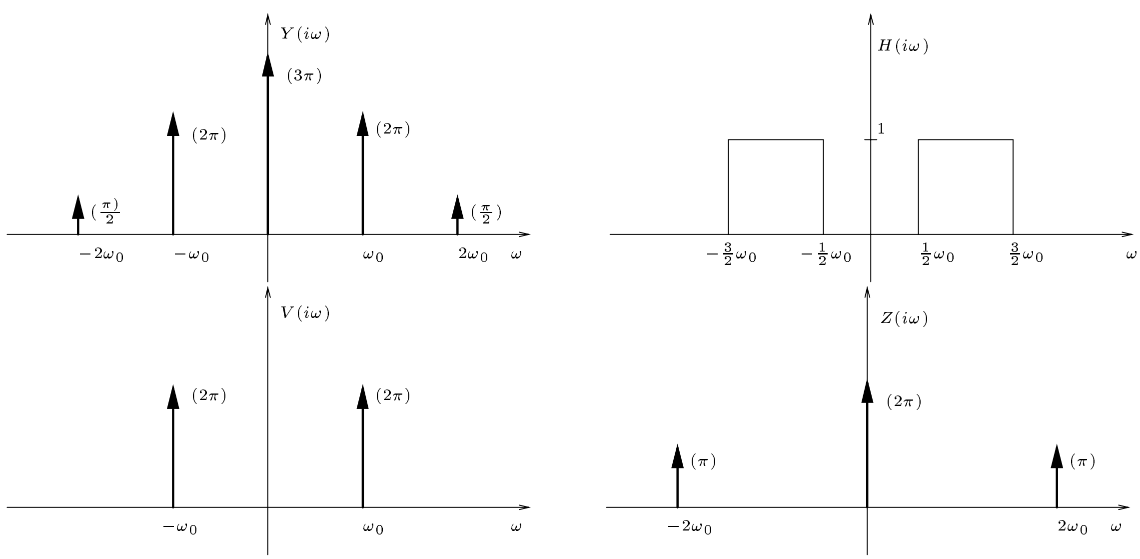
1.

$$\begin{aligned} y(t) &= (1+\cos(\omega_0 t))^2 = 1 + 2\cos(\omega_0 t) + \cos^2(\omega_0 t) \\ &= 1 + 2\cos(\omega_0 t) + \frac{1+\cos(2\omega_0 t)}{2} \\ &= \frac{3}{2} + 2\cos(\omega_0 t) + \frac{1}{2}\cos(2\omega_0 t) \end{aligned}$$

We can also find $y(t)$ using the exponential formula:

$$\begin{aligned} y(t) &= (1 + \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t})^2 \\ &= 1 + \frac{1}{4}e^{i2\omega_0 t} + \frac{1}{4}e^{-i2\omega_0 t} + e^{i\omega_0 t} + e^{-i\omega_0 t} + \frac{1}{2} \\ &= \frac{3}{2} + e^{i\omega_0 t} + e^{-i\omega_0 t} + \frac{e^{i2\omega_0 t} + e^{-i2\omega_0 t}}{4} \end{aligned}$$




$$v(t) = 2\cos(\omega_0 t)$$
$$z(t) = 1 + \cos(2\omega_0 t)$$

2. (a) If the Fourier Transform of $f(t)$ is defined as $F(\omega) = \mathcal{F}[f(t)]$, then from the duality property we know that

$$\mathcal{F}[F(t)] = 2\pi f(-\omega).$$

From the tables we have

$$\mathcal{F}[e^{-t}u(t)] = \frac{1}{1+i\omega}.$$

Using the duality property on this equation we can write

$$\mathcal{F}[\frac{1}{1+it}] = 2\pi e^{\omega}u(-\omega).$$

If we use the fact that $\mathcal{F}[g(-t)] = G(-\omega)$ we get the final result

$$\mathcal{F}[\frac{1}{1-it}] = 2\pi e^{-\omega}u(\omega).$$

(b) For the product

$$f(t) = \frac{\text{sinc}(t)}{\pi} \cdot \frac{1}{1-it} = f_1(t).f_2(t)$$

we can write its Fourier Transform as a convolution

$$F(\omega) = \frac{1}{2\pi}F_1(i\omega) * F_2(i\omega) = \frac{1}{2\pi}\left[u(\omega+1) - u(\omega-1)\right] * 2\pi e^{-\omega}u(\omega).$$

Writing the convolution as an integral gives

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[u(\omega-\sigma+1) - u(\omega-\sigma-1)\right]2\pi e^{-\sigma}u(\sigma)d\sigma \\
&= \begin{cases} \int_{\omega-1}^{\omega+1} e^{-\sigma}d\sigma & \text{if } \omega > 1 \\ \int_{0}^{\omega+1} e^{-\sigma}d\sigma & \text{if } -1 \le \omega \le 1 \\ 0 & \text{if } \omega < -1 \end{cases} \\
&= \begin{cases} e^{-(\omega-1)} - e^{-(\omega+1)} & \text{if } \omega > 1 \\ 1 - e^{-(\omega+1)} & \text{if } -1 \le \omega \le 1 \\ 0 & \text{if } \omega < -1 \end{cases} \\
&= \left[1 - e^{-(\omega+1)}\right]\left[u(\omega+1) - u(\omega-1)\right] + \left[e^{-(\omega-1)} - e^{-(\omega+1)}\right]u(\omega-1) \\
&= \left[1 - e^{-(\omega+1)}\right]u(\omega+1) - \left[1 - e^{-(\omega-1)}\right]u(\omega-1)
\end{aligned}$$

3. (a)

$$H(i\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{0}^{T} e^{-i\omega t}dt = \frac{1-e^{-i\omega T}}{i\omega}$$

Now,

$$H(i\omega) = e^{-i\omega\frac{T}{2}}\left[\frac{e^{i\omega\frac{T}{2}} - e^{-i\omega\frac{T}{2}}}{i\omega}\right] = e^{-i\omega\frac{T}{2}}\left[\frac{2\sin(\frac{\omega T}{2})}{\omega}\right]$$

$H_R(\omega) = \frac{2\sin(\frac{\omega T}{2})}{\omega}$ is a real-valued sinc function. Its zero crossings are when

$$\frac{\omega T}{2} = n\pi$$

$$\text{or } \omega = n\omega_0, n = \pm 1, \pm 2, \ldots$$

$$\text{where } \omega_0 = \frac{2\pi}{T}$$

Therefore, $H(in\omega_0) = 0$ for $n \pm 1, \pm 2, \ldots$

(b)

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{in\omega_0 t}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} Y_n e^{in\omega_0 t} = \sum_{n=-\infty}^{+\infty} X_n H(in\omega_0) e^{in\omega_0 t}$$

Now, from (a),

$$H(in\omega_0) = 0 \text{ for } n \neq 0$$

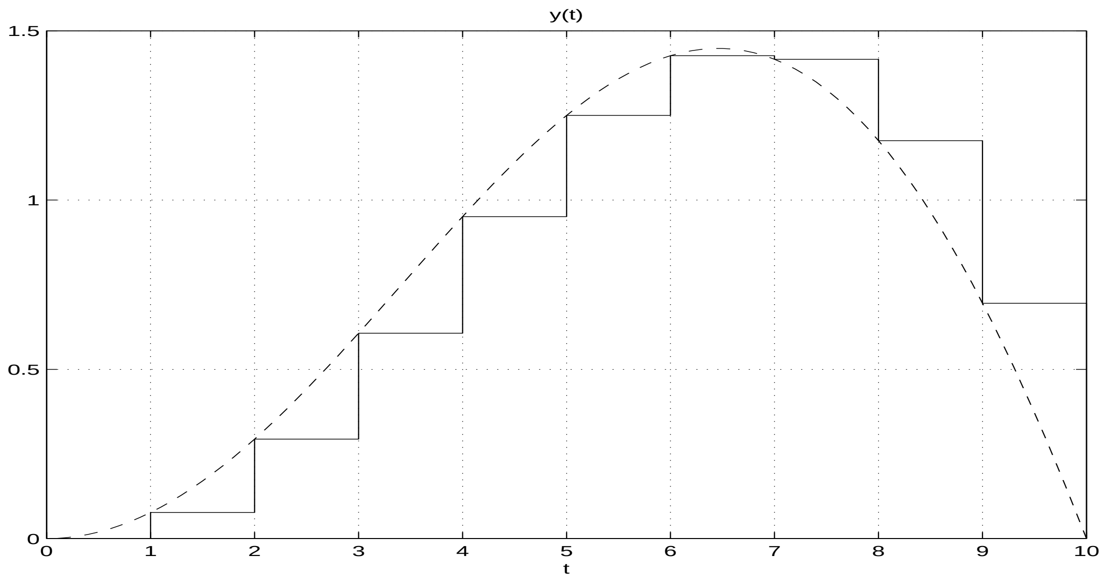
$$\text{and } H(0) = T$$

Therefore,

$$y(t) = TX_0, \text{ a constant function}$$

We can also see this in the time domain:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} [u(t-\tau) - u(t-\tau-T)]x(\tau)d\tau \\ &= \int_{t-T}^{t} x(\tau)d\tau = T.\frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau \\ &= TX_0 \text{ since } x(t) \text{ is periodic} \end{aligned}$$




(c)

$$x(t) = f(t) \sum_{n=-\infty}^{+\infty} \delta(t - kT) = \sum_{n=-\infty}^{+\infty} f(kT)\delta(t - kT)$$

by the property $f(t)\delta(t - kT) = f(kT)\delta(t - kT)$.
Since $\mathcal{S}$ is an LTI system,

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{+\infty} f(kT)h(t - kT) \\ &= \sum_{n=-\infty}^{+\infty} f(kT)[u(t - kT) - u(t - (k+1)T)] \end{aligned}$$

$y(t)$ is the staircase function in the figure (where $T = 1$).
We saw in class that

$$X(i\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} F(i(\omega - n\omega_0))$$

Therefore

$$Y(i\omega) = \frac{1 - e^{-i\omega T}}{i\omega T} \sum_{n=-\infty}^{+\infty} F(i(\omega - n\omega_0))$$