



Introduction to Linear Algebra

DEFINITION

- A "SYSTEM" OF EQUATIONS IS A SET OR COLLECTION OF EQUATIONS THAT YOU DEAL WITH ALL TOGETHER AT ONCE. LINEAR EQUATIONS (ONES THAT GRAPH AS STRAIGHT LINES) ARE SIMPLER THAN NON-LINEAR EQUATIONS, AND THE SIMPLEST LINEAR SYSTEM IS ONE WITH TWO EQUATIONS AND TWO VARIABLES.

LINEAR EQUATION

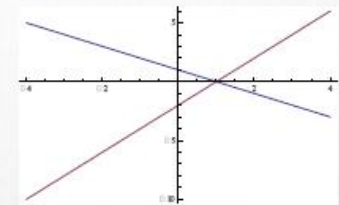
- A SYSTEM OF LINEAR EQUATIONS HAS
 1. NO SOLUTION, OR
 2. EXACTLY ONE SOLUTION, OR
 3. INFINITELY MANY SOLUTIONS.
- A SYSTEM OF LINEAR EQUATIONS IS SAID TO BE **CONSISTENT** IF IT HAS EITHER ONE SOLUTION OR INFINITELY MANY SOLUTIONS.
- A SYSTEM OF LINEAR EQUATION IS SAID TO BE **INCONSISTENT** IF IT HAS NO SOLUTION.

SOLUTION SET

Unique Solution

These two lines intersect in a single point.

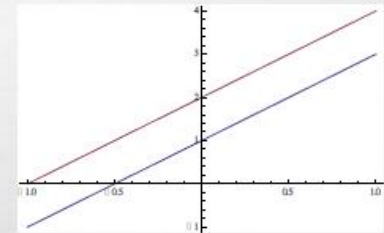
$$\begin{aligned}x + y &= 1 \\ 2x - y &= 2\end{aligned}$$



No Solution

Here the lines are parallel, so never intersect.
In this case we call the system inconsistent.

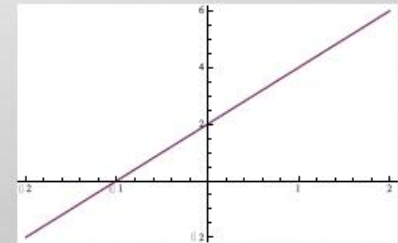
$$\begin{aligned}4x - 2y &= 2 \\ 2x - y &= 2\end{aligned}$$



Infinitely many solutions

The two lines coincide in this case, so they have an infinite number of intersection points.

$$\begin{aligned}4x - 2y &= 4 \\ 2x - y &= 2\end{aligned}$$



MATRIX NOTATION

- THE ESSENTIAL INFORMATION OF A LINEAR SYSTEM CAN BE RECORDED COMPACTLY IN A RECTANGULAR ARRAY CALLED A MATRIX.
- FOR THE FOLLOWING SYSTEM OF EQUATIONS,

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9,$$

THE MATRIX

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

IS CALLED THE COEFFICIENT MATRIX OF THE SYSTEM.

AUGMENTED MATRIX

- AN AUGMENTED MATRIX OF A SYSTEM CONSISTS OF THE COEFFICIENT MATRIX WITH AN ADDED COLUMN CONTAINING THE CONSTANTS FROM THE RIGHT SIDES OF THE EQUATIONS.
- FOR THE GIVEN SYSTEM OF EQUATIONS,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

IS CALLED THE AUGMENTED MATRIX.

Basic concepts

- **Vector** in \mathbb{R}^n is an ordered set of n real numbers.

- e.g. $v = (1,6,3,4)$ is in \mathbb{R}^4

- A column vector:

$$\begin{pmatrix} 1 \\ 6 \\ 3 \\ 4 \end{pmatrix}$$

- A row vector:

$$(1 \ 6 \ 3 \ 4)$$

- m -by- n **matrix** is an object in $\mathbb{R}^{m \times n}$ with m rows and n columns, each entry filled with a (typically) real number:

$$\begin{pmatrix} 1 & 2 & 8 \\ 4 & 78 & 6 \\ 9 & 3 & 2 \end{pmatrix}$$

Basic concepts

Vector norms: A norm of a vector $\|x\|$ is informally a measure of the “length” of the vector.

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

– Common norms: L_1 , L_2 (Euclidean)

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

– L_{∞}

$$\|x\|_{\infty} = \max_i |x_i|$$

Basic concepts

We will use lower case letters for vectors The elements are referred by x_i .

- **Vector dot (inner) product:**

$$x^T y \in \mathbb{R} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} y_1 \\ x_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i.$$

If $u \cdot v = 0$, $\|u\|_2 \neq 0$, $\|v\|_2 \neq 0 \rightarrow u$ and v are **orthogonal**

If $u \cdot v = 0$, $\|u\|_2 = 1$, $\|v\|_2 = 1 \rightarrow u$ and v are **orthonormal**

- **Vector outer product:**

$$xy^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} [y_1 \quad y_2 \quad \cdots \quad y_n] = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix}$$

Special Matrices

- Matrices where $m=n$ are called *square matrices*.
- There are a number of special forms of square matrices:

Symmetric $[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$	Diagonal $[A] = \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$	Identity $[A] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$
Upper Triangular $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix}$	Lower Triangular $[A] = \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	Banded $[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$

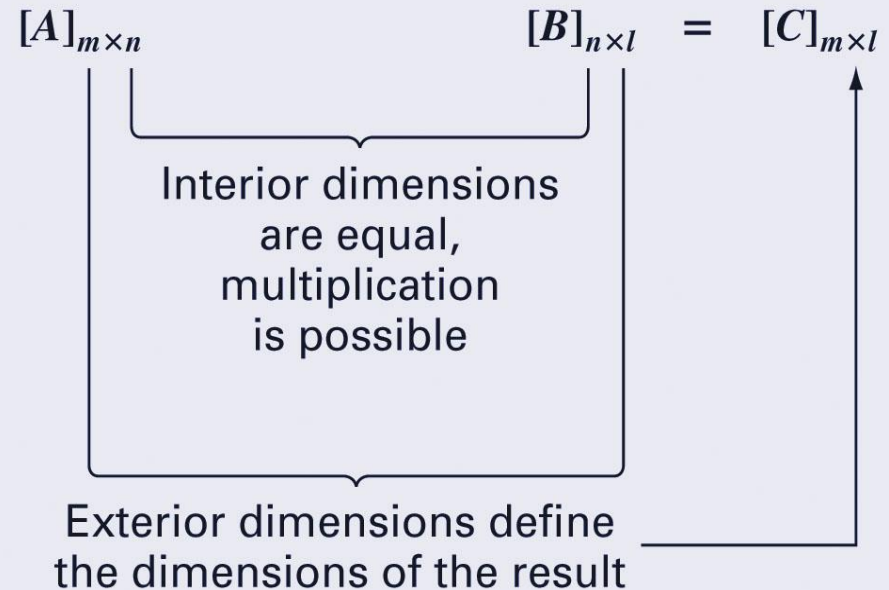
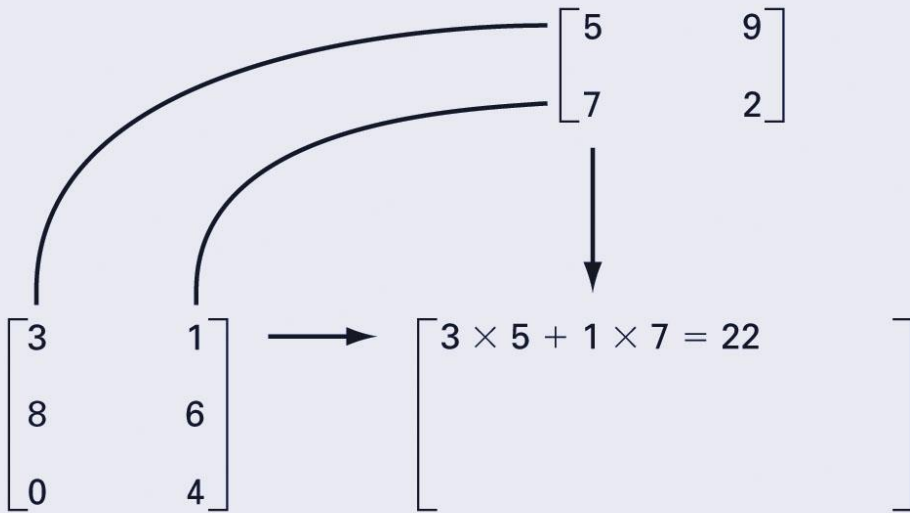
Matrix Operations

- Two matrices are considered equal if and only if every element in the first matrix is equal to every corresponding element in the second. This means the two matrices must be the same size.
- Matrix addition and subtraction are performed by adding or subtracting the corresponding elements. This requires that the two matrices be the same size.
- Scalar matrix multiplication is performed by multiplying each element by the same scalar.

Matrix Multiplication

- The elements in the matrix $[C]$ that results from multiplying matrices $[A]$ and $[B]$ are calculated using:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$



Basic concepts

We will use upper case letters for matrices. The elements are referred by $A_{i,j}$.

- **Matrix product:**

$$A \in \mathbb{R}^{m \times n} \quad B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Matrix Inverse and Transpose

- The *inverse* of a square, nonsingular matrix $[A]$ is that matrix which, when multiplied by $[A]$, yields the identity matrix.

$$[A][A]^{-1}=[A]^{-1}[A]=[I]$$

- The *transpose* of a matrix involves transforming its rows into columns and its columns into rows.

$$(a_{ij})^T=a_{ji}$$

Matrix Operations

- ***Matrix augmentation*** means adding column(s) to the original matrix
 - Some linear algebra techniques involve this
 - e.g. augment a matrix with an identity matrix with the same size
 - can do this by just concatenating the matrices together using square brackets []

Matrix Inverse

- If the result of multiplying a matrix A by a matrix B is the identity matrix, then B is the *inverse* of matrix A
- The inverse of a matrix A is written as A^{-1}
- So, $[A][A^{-1}] = [I]$
- How to actually compute the inverse A^{-1} of a matrix by hand is not so easy; we will look at some techniques next
- MATLAB, however, has a function **inv** to compute a matrix inverse

Basic concepts

Transpose: You can think of it as

- “flipping” the rows and columns

OR

- “reflecting” vector/matrix on line

e.g. $\begin{pmatrix} a \\ b \end{pmatrix}^T = (a \ b)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- $(A^T)^T = A$

- $(AB)^T = B^T A^T$

- $(A + B)^T = A^T + B^T$

Rank of a Matrix

- $\text{rank}(A)$ (the rank of a m -by- n matrix A) is
 - The maximal number of linearly independent columns
 - =The maximal number of linearly independent rows
 - =The dimension of $\text{col}(A)$
 - =The dimension of $\text{row}(A)$
- If A is n by m , then
 - $\text{rank}(A) \leq \min(m, n)$
 - If $n = \text{rank}(A)$, then A has full row rank
 - If $m = \text{rank}(A)$, then A has full column rank

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

Inverse of a matrix

- Inverse of a square matrix A , denoted by A^{-1} is the *unique* matrix s.t.
 - $AA^{-1}=A^{-1}A=I$ (identity matrix)
- If A^{-1} and B^{-1} exist, then
 - $(AB)^{-1} = B^{-1}A^{-1}$,
 - $(A^T)^{-1} = (A^{-1})^T$
- For orthonormal matrices
- For diagonal matrices

$$\mathbf{A}^{-1} = \mathbf{A}^T$$

$$\mathbf{D}^{-1} = \text{diag}\{d_1^{-1}, \dots, d_n^{-1}\}$$

commands

Transpose	$B = A'$
Identity Matrix	$\text{eye}(n)$ → returns an $n \times n$ identity matrix $\text{eye}(m,n)$ → returns an $m \times n$ matrix with ones on the main diagonal and zeros elsewhere.
Addition and subtraction	$C = A + B$ $C = A - B$
Scalar Multiplication	$B = \alpha A$, where α is a scalar.
Matrix Multiplication	$C = A * B$
Matrix Inverse	$B = \text{inv}(A)$, A must be a square matrix in this case. $\text{rank}(A)$ → returns the rank of the matrix A .
Matrix Powers	$B = A.^2$ → squares each element in the matrix $C = A * A$ → computes $A * A$, and A must be a square matrix.
Determinant	$\text{det}(A)$, and A must be a square matrix.

A, B, C are matrices, and m, n, α are scalars.

Linear Algebraic Equations

- A *system S* of linear algebraic equations is a set of equations of the form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

...

...

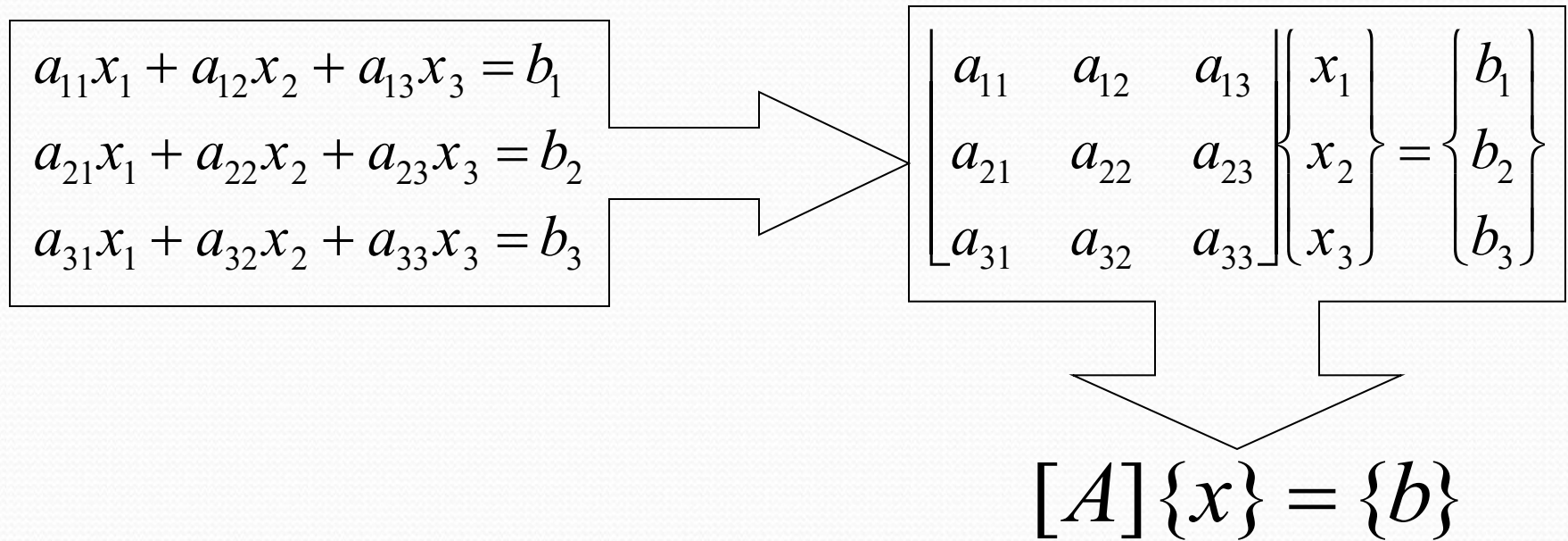
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$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

- This is called an $m \times n$ system of equations; there are m equations and n unknowns.

Representing Linear Algebra

- Matrices provide a concise notation for representing and solving simultaneous linear equations:



Solving the Matrix Form

- All systems of linear equations have either:
 - no solutions
 - one solution
 - infinitely many solutions
- To solve for x :
$$x = A^{-1} b$$
- So, if we can find the matrix inverse A^{-1} , we can solve!
(Again, this uses matrix multiplication; it says that the column vector x is the inverse of A multiplied by the column vector b)

If a solution exists, how do we know whether it is unique?

In a system of equations $[A][X] = [C]$ that is consistent, the rank of the coefficient matrix $[A]$ is the same as the augmented matrix $[A|C]$. If in addition, the rank of the coefficient matrix $[A]$ is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix $[A]$ is less than the number of unknowns, then infinite solutions exist.

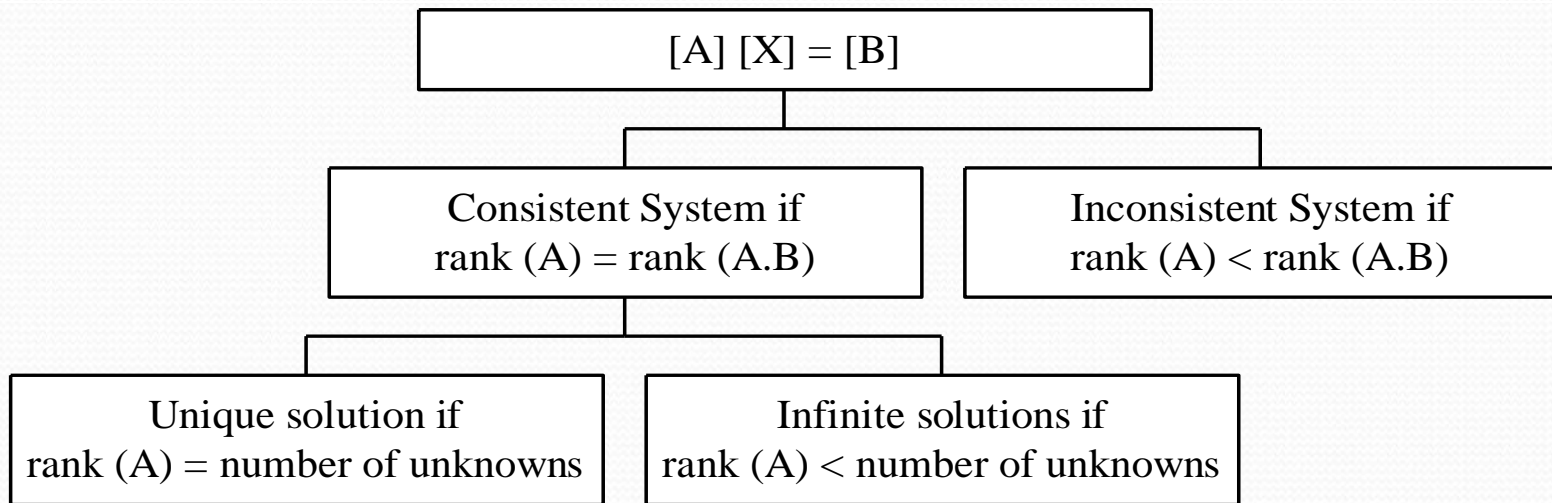


Figure 5.2. Flow chart of conditions for consistent and inconsistent system of equations.

Solving With MATLAB

- MATLAB provides two direct ways to solve systems of linear algebraic equations $[A]\{x\}=\{b\}$:
 - Left-division
 $x = A \backslash b$
 - Matrix inversion
 $x = \text{inv}(A) * b$
- The matrix inverse is less efficient than left-division and also only works for square, non-singular systems.

Solutions to Systems of Linear Equations

- Example: a system of 3 linear equations with 3 unknowns (x_1, x_2, x_3):

$$3x_1 + 2x_2 - x_3 = 10$$

$$-x_1 + 3x_2 + 2x_3 = 5$$

$$x_1 - x_2 - x_3 = -1$$

Let :

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 5 \\ -1 \end{bmatrix}$$

Then, the system can be described as:

$$Ax = b$$

Solutions to Systems of Linear Equations (con't...)

- Solution by Matrix Inverse:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

- MATLAB:

```
>> A = [ 3 2 -1; -1 3 2; 1 -1 -1];
```

```
>> b = [ 10; 5; -1];
```

```
>> x = inv(A)*b
```

```
x =
```

```
-2.0000
```

```
5.0000
```

```
-6.0000
```

Answer:

$$x_1 = -2, x_2 = 5, x_3 = -6$$

- Solution by Matrix Division:

The solution to the equation

$$Ax = b$$

can be computed using **left division**.

- MATLAB:

```
>> A = [ 3 2 -1; -1 3 2; 1 -1 -1];
```

```
>> b = [ 10; 5; -1];
```

```
>> x = A\b
```

```
x =
```

```
-2.0000
```

```
5.0000
```

```
-6.0000
```

Answer:

$$x_1 = -2, x_2 = 5, x_3 = -6$$

NOTE:

left division: $A \setminus b \rightarrow b \div A$

right division: $x / y \rightarrow x \div y$

2 x 2 Systems

For a 2 x 2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

the matrix inverse is defined as

$$A^{-1} = \frac{1}{D} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where the D is called the determinant and is defined as:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

MATLAB Functions

- Related functions in MATLAB:
 - matrix inverse: **inv**
 - determinant: **det**
 - reduce to reduced row echelon form: **rref**
- To solve systems of equations in MATLAB
 - $x = \text{inv}(A)*b$
 - $x = A \setminus b$

Symbolic Expressions

- Create symbolic variables using the `sym` function, e.g.
 - `a = sym('a');`
 - Shortcut for a lot of these: `syms x y z`
 - `symvar = sym('x^3 - 2');`
- Symbolic math: doing math on symbols!
 - Using normal operators e.g. `+`, `-`, `*`, etc.
- Symbolic expressions are rational, e.g. kept in fractional form so `sym(2/4)` returns `1/2` rather than `0.5`

Symbolic Functions

- **simplify** simplifies expressions
- **collect** collects like terms
- **expand** multiplies out terms
- **factor** factors a symbolic expression
- **subs** substitutes a value into an expression
- **numden** returns separately the numerator and denominator of a fraction
- **pretty** is a display function; shows exponents
- **ezplot** will draw a 2-D plot in the x-range from -2π to 2π

Examples

```
>> y = sym('y');  
>> a = y * sym('y^2')  
a =  
y^3
```

```
>> a/y  
ans =  
y^2
```

```
>> subs(a,4)  
ans =  
64
```

```
>> 1/4 + 3/6  
ans =  
0.7500
```

```
>> [n d] = numden(sym(1/4 + 3/6))  
n =  
3  
  
d =  
4
```

```
>> syms a  
>> expand((a+3)*(a-2))  
ans =  
a^2+a-6
```

Polynomial/Symbolic Conversions

- **sym2poly(s)** converts from a symbolic expression s to a row vector representing polynomial coefficients
- **poly2sym(p)** converts from the row vector representing polynomial coefficients p to a symbolic expression

Solving Equations

- The **solve** function can solve a single equation or a set of equations
- Part of Symbolic Math Toolbox
- For a single equation,
 - will return symbolic expressions (**double** can convert)
 - will determine for which variable to solve (closest to x)
- Sets of equations:
 - results are symbolic expressions stored in a structure; the variable names are the names of the fields

Consistent and inconsistent system

A system of equations can be consistent or inconsistent. What does that mean?

A system of equations $[A][X]=[C]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, a consistent system of equations does not mean a unique solution, that is, a consistent system of equations may have a unique solution or infinite solutions (Figure 1).

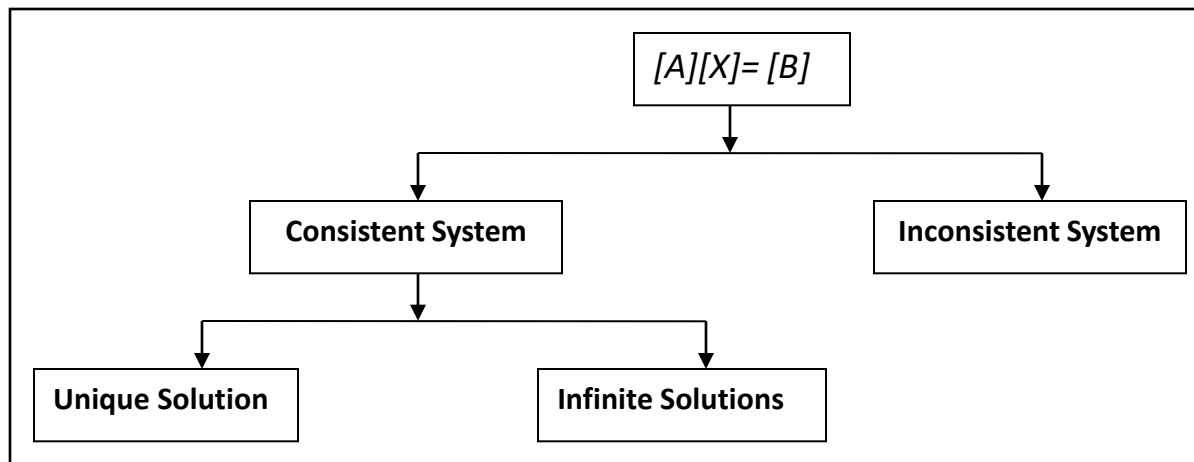


Figure 5.1. Consistent and inconsistent system of equations flow chart.

Example 2

Give examples of consistent and inconsistent system of equations.

Solution

a) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is a consistent system of equations as it has a unique solution, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 2 (cont.)

b) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

is also a consistent system of equations but it has infinite solutions as given as follows. Expanding the above set of equations,

$$2x + 4y = 6$$

$$x + 2y = 3$$

Example 2 (cont.)

you can see that they are the same equation. Hence, any combination of (x, y) that satisfies,

$$2x + 4y = 6$$

is a solution. For example $(x, y) = (1, 1)$ is a solution. Other solutions include

$(x, y) = (0.5, 1.25)$, $(x, y) = (0, 1.5)$, and so on.

c) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is inconsistent as no solutions exists.

Distinguishing consistency

How can one distinguish between a consistent and inconsistent system of equations?

A system of equations $[A][X]=[C]$ is *consistent* if the rank of A is equal to the rank of the augmented matrix $[A:C]$.

A system of equations $[A][X]=[C]$ is *inconsistent* if the rank of A is less than the rank of the augmented matrix $[A:C]$.

But, what do you mean by rank of a matrix?

The rank of a matrix is defined as the order of the largest square submatrix whose determinant is not zero.