EECE 360
Lecture 5

State-Space Descriptions

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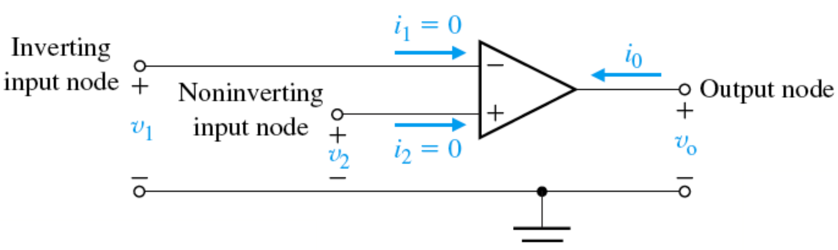
Chapter 2.9-2.10, 3.1-3.3

Outline

- Last class
  - Op-Amps; PID
  - Block diagram reductions
- Today
  - State differential equations
  - State-space representation
  - Transfer function of state-space form
- Next time
  - Linear Algebra needed to solve state-space eq'ns

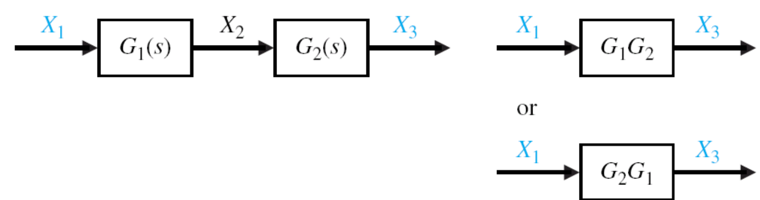
Review: Op-Amps

- Ideal operating conditions:
  - $i_1=0$ and $i_2=0$ (input impedance is infinite)
  - $v_1=v_2$



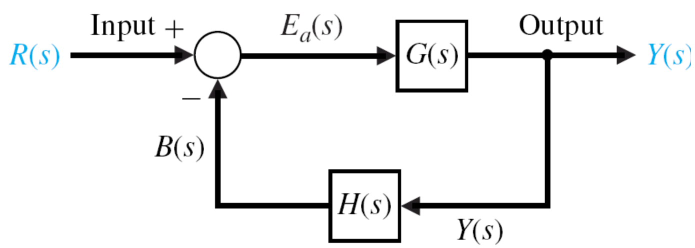
Block Diagram Transformations

- Combining blocks in cascade: multiplication is commutative
- **When combining blocks, the input-output relationship (transfer function) should not change.**



$$X_3 = G_2X_2 = G_2(G_1X_1) = G_2G_1X_1$$

Closed-Loop Transfer Function**



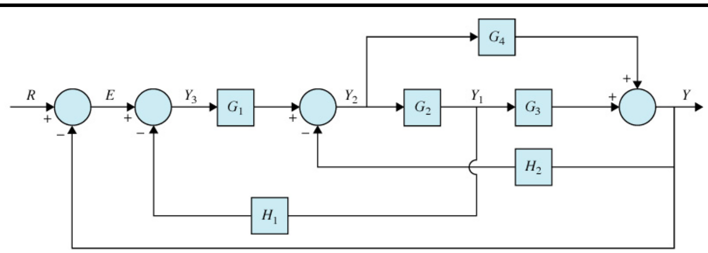
$$\frac{Y(s)}{R(S)} = \frac{G(s)}{1 + G(s)H(s)}$$

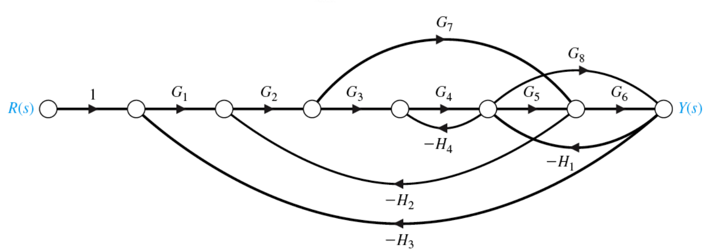
Extremely important!

Signal Flow Graphs (SFGs)

- Alternative to block diagrams
- Block diagrams require iterative reduction to compute overall transfer function
- With SFG's, possible to compute overall transfer function using a formula, Mason's gain formula
- Dorf and Bishop, Chapter 2.7

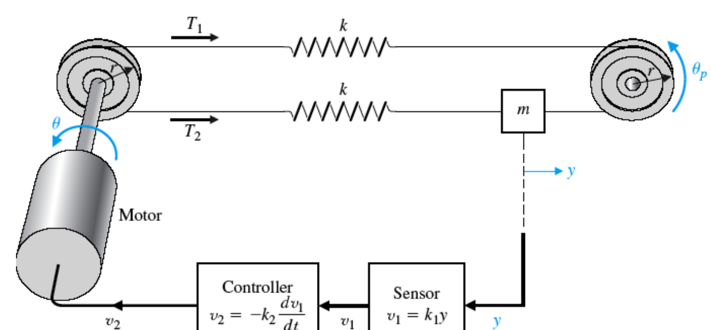
Signal Flow Graphs





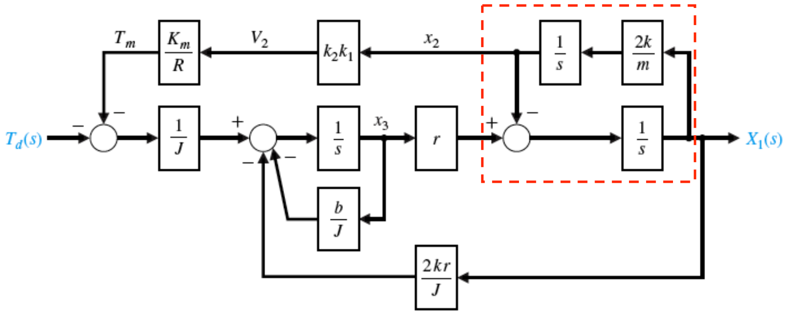
Example: Printer belt-drive

- Belt drive moves printing device
- Light sensor measures position of printing device
- Motor (actuator) adjusts belt drive



Example: Printer belt-drive

- Transfer function between
  - Output = position of printing device
  - Input = disturbance torques acting on the motor
- How can this be reduced?



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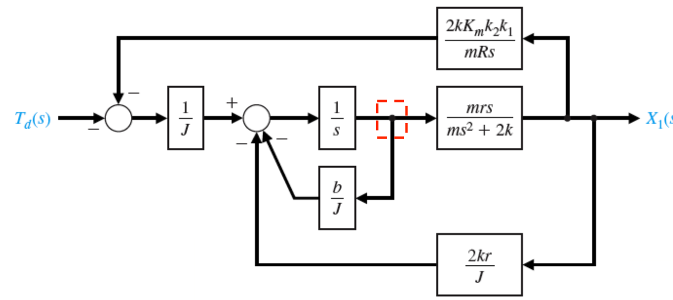

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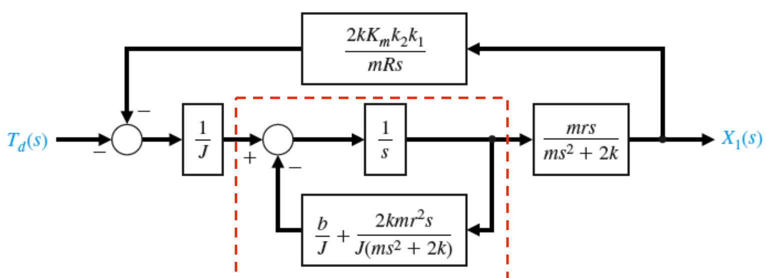

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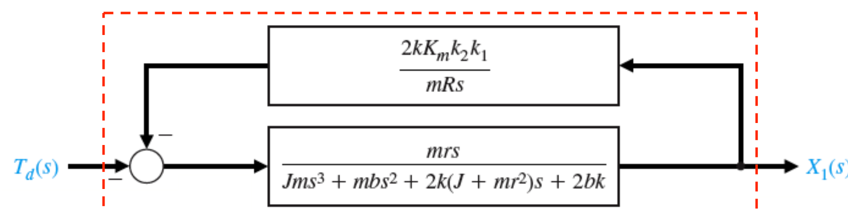
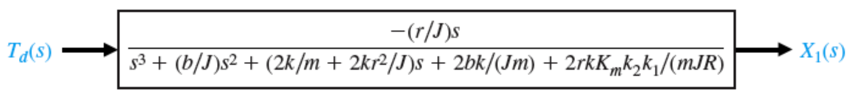


Example: Printer belt-drive

- Transfer function between
  - Output = position of printing device
  - Input = disturbance torques acting on the motor
- How can this be reduced?
- Notes:
  - More than one way to achieve this reduction
  - Could also solve this algebraically from 1st diagram

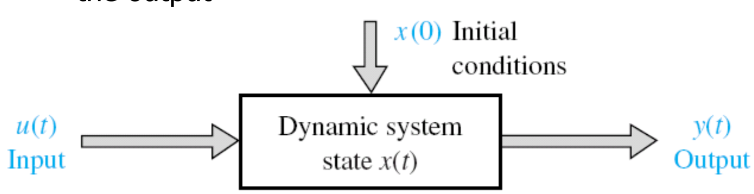


State equations

- State differential equations are an alternative way to describe a dynamic system (i.e. time-domain method)
- For LTI systems, it is routine to move between state-space and transfer function representations (i.e. between time-domain and frequency domain)
- Most multivariable design methods are based on state equations
- Basic technology: A physical system can be described by a high order differential equation

A General Dynamic System

- The state of a system is described in terms of state variables $[x_1(t), x_2(t), x_3(t), \cdots, x_n(t)]$
- Given the input *u(t)*, initial conditions $x_0(t)$, and the present state *x(t)*, we can determine
  - the future system behavior
  - the output



State Variables of a Dynamic System

- "The state of a system is a set of variables such that the knowledge of these variables and of the input function will, with the equations describing the dynamics, provide the future state and output of the system"
- "The state variables describe the future response of a system, given the present state, the excitation inputs and the equations of the system"

The Concept of State

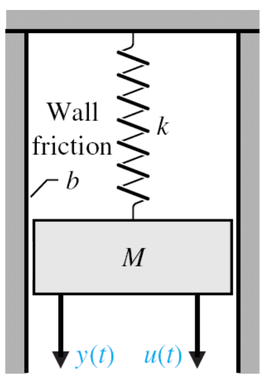
- Not new!
- Tycho Brahe, Kepler, Newton,…
- To predict the future motion of a planet, it is enough to know its current position and velocity as well as the equations governing its motion.
- Russian control school: Lyapunov, Pontryagin, etc. always worked in time domain
- The state is the minimum information required about a system to predict its future

The Concept of State

- The state gives a complete description of the system at a given time *t*, i.e. captures the evolution of the system up to time *t*
- This implies that there are orderly rules for transition from one state to another
- **State-space descriptions are not unique** – but input/output relations stay the same

Example: Spring-Mass-Damper

- Previously, we found
  $$u(t) = M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t)$$
- Now define
  $$x_1(t) = y(t), x_2(t) = \frac{dy(t)}{dt}$$
- such that
  $$u(t) = M\frac{dx_2(t)}{dt} + bx_2(t) + kx_1(t)$$



** - Write as 2 first-order diff. eqns.

$$\begin{aligned}\frac{dx_1(t)}{dt} &= x_2(t)\\ \frac{dx_2(t)}{dt} &= -\frac{b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t)\end{aligned}$$

The Concept of State

- Typically for a physical system, the state relates to the concept of energy – each state describes an energy storage component (mass, spring, inductor, capacitor etc.)
- Often it is useful to use variables that are readily measured, e.g. currents, voltages, positions, velocities, pressures, temperatures, concentrations, etc…
- State vector *x(t)*:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \qquad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix};$$

State-Space Description

- The state-space description leads to a system of coupled, first-order differential equations
- Reduction to first-order is a general technique to transform a high-order differential equation into a state-space model

$$y(t) = a_0 f(t) + a_1 f'(t) + a_2 f''(t) + \cdots a_n f^{(n)}(t)$$

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ &\vdots \\ x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{aligned}$$

State-Space Description

- Write in matrix form

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ &\vdots \\ x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{aligned}$$

$$\dot{x} = Ax, \quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The State Differential Equation

- The general state space description for a *linear time-invariant, continuous-time* dynamical system is:

system matrix, input matrix, input vector

State differential equation: $\dot{x}(t) = Ax(t) + Bu(t)$ (1)

Output equation: $y(t) = Cx(t) + Du(t)$ (2)

output matrix

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$.

- A is $(n \times n)$, B is $(n \times m$, C is $(p \times n)$ and D is $(p \times m)$. Shorthand for this system is $[A, B, C, D]$

Example: RLC Circuit

- Consider an RLC circuit described by:

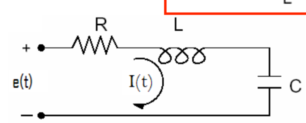
$$e(t) = RI(t) + L\frac{d}{dt}I(t) + \frac{1}{C}\int_0^t I(\tau)d\tau$$

With $x_1(t) = \int_0^t I(\tau)d\tau$ and $x_2(t) = I(t)$ the state-space formulation is

state vector

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e(t)$$

State differential equation



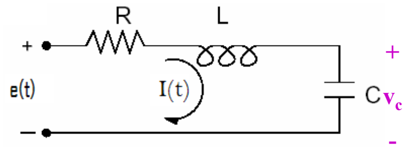
$$I(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Output equation

output

Example: RLC Circuit

- State-space representations are NOT unique



$$i = x_1 \qquad v_c = x_2$$

$$i = C\dot{v}_c \qquad \dot{x}_2 = \frac{1}{C}x_1$$

$$e = iR + L\frac{di}{dt} + v_c$$

$$e = Rx_1 + L\dot{x}_1 + x_2$$

$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}e$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & -\frac{1}{L}\\ \frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}e$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

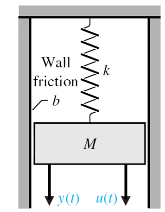
Example: Spring-Mass-Damper

- First-order differential equations

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t)$$

- In matrix form:

$$\begin{bmatrix}\frac{dx_1(t)}{dt}\\ \frac{dx_2(t)}{dt}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}x(t)$$



- with

$$A = \begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix}, \quad B = \begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}, \quad C = \begin{bmatrix}1 & 0\end{bmatrix}, \quad D = 0$$

Solution to the State Eq'ns

- Why is this representation so useful? Because we know what its solution looks like.

- Given the state equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- We can show that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Note that this is a **matrix exponential!

Transfer function of state eqns

- Take the Laplace transform of

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Re-arrange to find *X(s)* in terms of *U(s), x(0):*

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)X(s) = BU(s) + x(0)$$
$$\text{So,} \quad X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)$$

Transfer function of state eqns

- Assume $x(0)=0$ for simplicity.
- Then substituting

$$X(s) = (sI - A)^{-1} BU(s)$$

into the output equation

$$Y(s) = CX(s) + DU(s)$$

yields

$$\boxed{Y(s) = (C(sI - A)^{-1} B + D)U(s) = G(s)U(s)}$$

- For SISO case, the **transfer function** is

$$G(s) = Y(s)/U(s)$$

Summary

- Reduction from n^{th} order differential equation to n first order differential equations
- State-space representations are NOT UNIQUE
- Known closed-form solution
- Transfer function for SISO state-space description

- Next time:
  - Solving the state-space equations
  - Linear algebra review