EECE 360
Lecture 6

State Equation Representation of Dynamic Systems

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Chapter 3.1 - 3.5

Outline

- Last class
  - Transfer functions vs. state-space models
  - Creating state-space models from n^{th} order differential equations
- Today and next class
  - Review and context
  - State-space models --> transfer functions
  - Linear algebra review
  - Closed-form solution to state-space models

Context: What we've done

- Introduction to control systems
  - Actuators, sensors, and the role of control
- Modeling of control systems
  - Time domain (F=ma, KVL, KCL, etc.)
    - Linearization
  - Frequency domain (transfer functions)
    - Laplace transform
    - Block diagram manipulation
    - Implementation through op-amps

Context: What we're doing

- State-space models
  - **Specific** time-domain model that is very useful
  - For the purpose of control, will be a **complementary but related framework** to frequency domain methods

  - From state-space to transfer function
  - From transfer function to state-space
  - Closed-form solution ($x(t)$ = …)
  - Requires some linear algebra

Context: What we will do

- Feedback characteristics
  - What the closed-loop systems should look like
- Control in frequency domain *(classical control)*
  - Root locus
  - Bode diagrams
  - Nyquist criterion
- Control in state-space domain *(modern control)*
  - Pole placement
  - Controllability
  - Observability

Context: General picture

- How LTI systems can and should behave
  - Modeling of LTI systems
    - Frequency domain (transfer functions)
    - Time domain (state-space descriptions)
  - Tools to analyze LTI systems
    - Frequency domain
    - Time domain
- How to design controllers to make LTI systems behave in a desired manner
  - Frequency domain
  - Time-domain

State-Space Models

Transfer function $\xleftrightarrow{\text{Laplace transformation}}$ State-space equations

- The state-variable model, state-space model, or state-space description
  - A differential equation model
  - Equations are written in a *specific* format
  - Expressed as *n* first-order coupled differential equations
  - Preserve the system's input-output relationship (for the same transfer function)

State-space models

- State differential equations are an alternative way to describe a dynamic system (*time-domain* method)
- For LTI systems, it is routine to move between state and transfer function representations i.e. between frequency and time domains
- Most multivariable design methods are based on state equations
- Basic technology: A physical system can be described by a high order differential equation,

One n^{th} order differential equation $\xrightarrow{\text{State-space modeling}}$ Set of n 1^{st} order differential equations

The State Differential Equation

- The general state space description for a *linear time-invariant, continuous-time* dynamical system is:

system matrix · input matrix · input vector

State differential equation:
$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

Output equation:
$$y(t) = Cx(t) + Du(t) \quad (2)$$

output matrix

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$.

- A is $(n \times n)$, B is $(n \times m$, C is $(p \times n)$ and D is $(p \times m)$. $\boxed{\text{Shorthand for this system is } [A, B, C, D]}$

State-space models

- To get a state-space model:
  - Start with a high-order differential equation
  - Convert to a set of 1st order coupled differential equations
  - Write in state-space form (A,B,C,D)

Example: RLC Circuit

- Consider an RLC circuit described by:

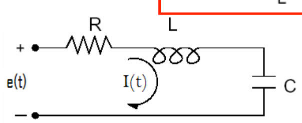
$$e(t) = RI(t) + L\frac{d}{dt}I(t) + \frac{1}{C}\int_0^t I(\tau)d\tau$$

With $\boxed{x_1(t) = \int_0^t I(\tau)d\tau \text{ and } x_2(t) = I(t)}$ the state-space formulation is

state vector

$$\boxed{\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e(t)}$$

State differential equation



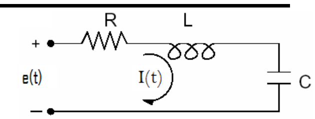
$$I(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Output equation

output

Example: RLC Circuit

- Consider the RLC circuit:
- Applying KVL yields



$$e(t) = RI(t) + L\frac{d}{dt}I(t) + \frac{1}{C}\int_0^t I(\tau)d\tau$$

- With input *e(t)* and output *I(t)*
- Choose the state x = [x_1(t), x_2(t)],

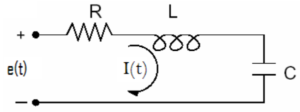
With $x_1(t) = \int_0^t I(\tau)d\tau$ and $x_2(t) = I(t)$

- Write the state differential equation in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

Example: RLC Circuit



- Through substitution:

$$\begin{aligned} e(t) &= Rx_2(t) + L\frac{dx_2}{dt} + \frac{1}{C}x_1(t) \\ \boxed{\dot{x}_2(t)} &\boxed{= -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}e(t)} \end{aligned}$$

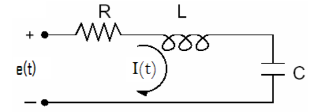
- Now recall that through definition of the state

$$\boxed{\dot{x}_1 = x_2(t)}$$

- Put coefficients into appropriate matrices

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Example: RLC Circuit



- Where

$$\begin{cases} A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \\ B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \\ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \\ D = 0 \end{cases}$$

- Represents the state-space model

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Example: 2^{nd} order system

- Consider the second-order system:

$$\ddot{y} + 2\omega\zeta\dot{y} + \omega^2 y = u(t)$$

With $\boxed{x = [\ y \quad \dot{y}\ ]^T}$ its state-space description is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\zeta \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\ 1 \quad 0\ ]x$$

State-space models

- Now that we have a state-space model:
  - How does this relate to transfer functions?
  - How can we find a transfer function from a given state-space model?

Transfer function of state eqns

- The Laplace transform of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

- (for a SISO system) is

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

- Re-arrange to find *X(s)* in terms of *U(s), x(0):*

$$(sI - A)X(s) = BU(s)$$
$$\text{So, } \boxed{X(s) = (sI - A)^{-1}BU(s)}$$

Transfer function of state eqns

- (Note: We assume *x(0)=0* for simplicity.)
- Then substituting

$$X(s) = (sI - A)^{-1}BU(s)$$

into the output equation

$$Y(s) = CX(s) + DU(s)$$

yields

$$\begin{aligned} Y(s) &= (C(sI - A)^{-1}B + D)U(s) \\ &= G(s)U(s) \end{aligned}$$

- For SISO case, the **transfer function** is**

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

Transfer function of state eqns

- Take the Laplace transform of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

- (with **non-zero** initial conditions)
- Re-arrange to find *X(s)* in terms of *U(s), x(0):*

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)X(s) = BU(s) + x(0)$$
$$\text{So, } \boxed{X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)}$$

Solution to the State Eq'ns

- Why is this representation so useful? Because we know what its solution looks like.
- Given the state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

- We can show that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

****Note that this is a matrix exponential!**

Key results: State-space to T.F.

- With zero initial conditions:

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

- With non-zero initial conditions:

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0) + DU(s)$$

- ** What is ***$(sI\text{-}A)^{-1}$***? We need some linear algebra to find this.