EECE 360

Lecture 6b

Linear Algebra Review #1

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Appendix E (online)

Linear Algebra Review

- To manipulate state-space representations of transfer functions, we need specific tools from linear algebra
- Matrix properties
- Matrix operations
- Matrix exponential...

See Appendix E from Dorf and Bishop.

Definitions

Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Element

Column vector

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Row vector

$$\mathbf{z} = [z_1 \quad z_2 \quad \cdots \quad z_n]$$

Diagonal matrix

$$\mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

i.e. $b_{ij} = 0, \text{ for } i \neq j$

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Symmetric matrix

$$\mathbf{H} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 6 & 4 \\ 1 & 4 & 8 \end{bmatrix}$$

i.e. $h_{ij} = h_{ji}$

Basic Operations

- Addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \rightarrow c_{ij} = a_{ij} + b_{ij}.$$

$$\boxed{\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.}$$

$$\boxed{(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).}$$

- Multiplication by a scalar

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{12} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

- Transpose

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.}$$

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 2 \\ 1 & 4 & 1 \\ -2 & 3 & -1 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 6 & 1 & -2 \\ 0 & 4 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

- Trace

$$\boxed{\operatorname{tr} \mathbf{A} = a_{11} + a_{22} + \ldots + a_{nn}.}$$

Basic Operations

- Matrix multiplication
  - For the matrix $\mathbf{C} = \mathbf{AB}$
  - Multiply rows of ***A*** by columns of ***B***

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iq}b_{qj} = \sum_{k=1}^{q} a_{ik}b_{kj}.$$

$$C_{21} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

  - Not commutative! $\boxed{\mathbf{AB} \neq \mathbf{BA}.}$

Basic Operations

$\boxed{\mathbf{AB} \neq \mathbf{BA}.}$

- Matrix multiplication
  - Inner dimensions must match

$$C = AB$$

$$n \times p \qquad n \times m \quad m \times p$$

  - Example

$$\mathbf{x} = \mathbf{Ay} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (a_{11}y_1 + a_{12}y_2 + a_{13}y_3) \\ (a_{21}y_1 + a_{22}y_2 + a_{23}y_3) \end{bmatrix}.$$

Sets of equations

- Matrix representation of linear equations

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= u_1, \\ 2x_1 + x_2 + 6x_3 &= u_2, \\ 4x_1 - x_2 + 2x_3 &= u_3. \end{aligned}$$

- State as a column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

- for which

$$\mathbf{Ax} = \mathbf{u},$$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 6 \\ 4 & -1 & 2 \end{bmatrix}.$$

Determinant of a Matrix

- Determinants are a measure of a matrix
- If determinant is zero, matrix is **singular**.
- 2 X 2 matrices:

$$\boxed{det\ \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.}$$

- *n* x *n* matrices:

$$\text{cofactor of } a_{ij} = \alpha_{ij} = (-1)^{i+j}M_{ij},$$

- M_{ij} is the determinant of the *(n-1)* x *(n-1)* matrix that results from removing row *i* and column *j*

$$M_{12} = \left| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \right|$$

Determinant, Adjoint

- *n* x *n* matrices:
  - Choose a row *i*: $det\ \mathbf{A} = \sum_{j=1}^{n} a_{ij}\alpha_{ij}$
  - Choose a column *j*: $det\ \mathbf{A} = \sum_{i=1}^{n} a_{ij}\alpha_{ij}$
- Example

$$det\ \mathbf{A} = \det \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}$$

$$= 2(-1) - (-5) + 2(3) = 9,$$

The Adjoint Matrix

- 2 x 2 matrices

$$\text{adjoint } \mathbf{A} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

- *n* x *n* matrices

$$adjoint\ \mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}^T = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix}$$

- Each term of the adjoint is a cofactor of A

Determinant Identities

- Multiplication
  - Single row or column by a constant k

$$\det \tilde{A} = k \det A$$

  - All elements of A by a constant k

$$\det(kA) = k^n \det A$$

- Transpose

$$\det A^T = \det A$$

- Matrix product (A, B square)

$$\begin{aligned} \det AB &= \det A \det B \\ \det AB &= \det BA \end{aligned}$$

Matrix Inversion

- Find A^{-1} such that for A square and |A|=0:

$$\begin{aligned} A^{-1}A &= I \\ AA^{-1} &= I \end{aligned}$$

- 2 x 2 matrix:

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- *n* x *n* matrix:

$$\boxed{\mathbf{A}^{-1} = \frac{\text{adjoint of } \mathbf{A}}{\det \mathbf{A}}}$$

Matrix Inversion

$$\boxed{\mathbf{A}^{-1} = \frac{\text{adjoint of } \mathbf{A}}{\det \mathbf{A}}}$$

- **Know formula for inverse of 2 x 2
- **Apply general formula to 3 x 3 matrices.

- Example:
  $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$
  - Find determinant
    $\det \mathbf{A} = -7.$
  - Find coefficients of the adjoint matrix (α_{11}, …, α_{33})

$$\alpha_{11} = (-1)^2 \begin{vmatrix} -1 & 4 \\ -1 & 1 \end{vmatrix} = 3.$$

$$\mathbf{A}^{-1} = \frac{\text{adjoint } \mathbf{A}}{\det \mathbf{A}} = \left(-\frac{1}{7}\right) \begin{bmatrix} 3 & -5 & 11 \\ -2 & 1 & 2 \\ -2 & 1 & -5 \end{bmatrix}.$$

Linear Algebra Review

- Basic matrix operations
- Matrix determinant

$$det\ \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- Matrix inverse

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Now back to the main topic:
  state-space --> transfer function