EECE 360

Lecture 7

State Equation Representation of Dynamic Systems

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Chapter 3.1 - 3.5

Outline

- Previously
  - Transfer functions vs. state-space models
- Today
  - Linear algebra review
  - State-space models --> transfer functions
  - Closed-form solution to state-space models
- Next time
  - Transfer function --> state-space models

State-space models

- To get a state-space model:
  - Start with a high-order differential equation
  - Convert to a set of 1st order coupled differential equations
  - Write in state-space form (A,B,C,D)
- Now that we have a state-space model:
  - How does this relate to transfer functions?
  - How can we find a transfer function from a given state-space model?

The State Differential Equation

- The general state space description for a *linear time-invariant, continuous-time* dynamical system is:

system matrix, input matrix, input vector

State differential equation: $$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

Output equation: $$y(t) = Cx(t) + Du(t) \quad (2)$$

output matrix

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$.

- A is $(n \times n)$, B is $(n \times m$, C is $(p \times n)$ and D is $(p \times m)$. Shorthand for this system is $[A, B, C, D]$

Key results: State-space to T.F.

- With zero initial conditions:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- With non-zero initial conditions:

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0) + DU(s)$$

- ** Need to find ***$(sI-A)^{-1}$***.

Linear Algebra Review

- To manipulate state-space representations of transfer functions, we need specific tools from linear algebra
- Matrix properties
- Matrix operations
- Matrix exponential...

See Appendix E (online) from Dorf and Bishop.

Matrix Inversion

$$\mathbf{A}^{-1} = \frac{\text{adjoint of } \mathbf{A}}{\det \mathbf{A}}$$

- Find A^{-1} such that

$$\begin{aligned} A^{-1}A &= I \\ AA^{-1} &= I \end{aligned}$$

- Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$

  - First find determinant

$$\det \mathbf{A} = -7.$$

  - Then find coefficients of the adjoint matrix (α_{11}, …, α

$$\alpha_{11} = (-1)^2 \begin{vmatrix} -1 & 4 \\ -1 & 1 \end{vmatrix} = 3.$$

$$\mathbf{A}^{-1} = \frac{\text{adjoint } \mathbf{A}}{\det \mathbf{A}} = \left(-\frac{1}{7}\right) \begin{bmatrix} 3 & -5 & 11 \\ -2 & 1 & 2 \\ -2 & 1 & -5 \end{bmatrix}.$$

Summary: Linear Algebra Rev.

- Basic matrix operations
- 2 x 2 matrix determinant

$$det\ \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- 2 x 2 matrix inverse

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- **Know specific formulas for 2 x 2 matrices.
- **Apply general formula to 3 x 3 matrices.

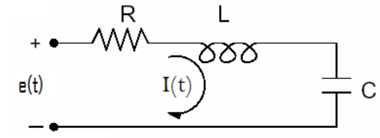
Summary: Linear Algebra Rev.

- *n* x *n* matrix determinant
  - For a row i $$det\ \mathbf{A} = \sum_{j=1}^{n} a_{ij}\alpha_{ij}$$
  - For a column j $$det\ \mathbf{A} = \sum_{i=1}^{n} a_{ij}\alpha_{ij}$$
- *n* x *n* matrix inverse

$$\mathbf{A}^{-1} = \frac{\text{adjoint of } \mathbf{A}}{\det \mathbf{A}}$$

- Now back to the main topic: **state-space --> transfer function**

Example: RLC Circuit



- Previously, with state x =[$x_1(t)$, $x_2(t)$],

$$x_1(t) = \int_0^t I(\tau)d\tau \text{ and } x_2(t) = I(t)$$

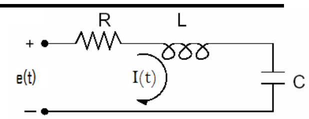
- we found the state-space equations

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e(t)$$

$$I(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

- **Now, find Y(s)/U(s) given (A,B,C,D)**

Example: RLC Circuit



$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

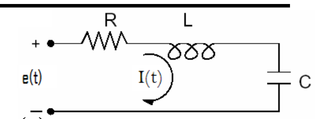
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{1}{LC} & s + \frac{R}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} + 0$$

$$= \frac{1}{s(s + \frac{R}{L}) + \frac{1}{LC}} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & 1 \\ -\frac{1}{LC} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

$$= \frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$G(s) = \frac{s}{Ls^2 + Rs + \frac{1}{C}}$$

Example: RLC Circuit Again



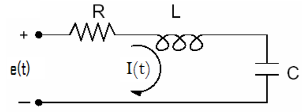
- Recall that with an alternate choice of state variables
  $$x_1(t) = I(t) \text{ and } x_2(t) = v_c(t)$$
  (keep input *e(t)* and output *I(t)* as before)
- We find an **equivalent** state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with different A,B,C matrices.

Example: RLC Circuit Again



$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} + 0$$

$$= \frac{1}{(s + \frac{R}{L})s + \frac{1}{LC}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{R}{L} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$= \frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$G(s) = \frac{s}{Ls^2 + Rs + \frac{1}{C}}$$

Note: This is the **same transfer function as before

Review: State-Space Models

- State-space descriptions are **not** unique – but input/output relations stay the same
  - An input-output transfer function can be represented by more than one SISO state-space model.
  - A SISO state-space model has only one transfer function associated with it.
- States are often variables that are readily measured, e.g. currents, voltages, positions, velocities, pressures, temperatures, concentrations, etc…

Relating state-space models

- One state-space model is related to another state-space model through a linear transformation *z = Px*.
- The matrix *P* must be *invertible* (the transformation must work in both directions).
- This is known as a **similarity transformation,** and the two state-space representations are said to be **similar**, or **equivalent**.
- We can show that transfer functions for both of these systems are the same.

Relating state-space models

- For a state-space model with state x

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- And transformation to a state z

$$\begin{aligned} z &= Px \\ P^{-1}z &= x \end{aligned}$$

$$\begin{aligned} \dot{z} &= P\dot{x} \\ &= PAx + PBu \\ &= \underbrace{PAP^{-1}}z + \underbrace{PB}u \end{aligned}$$

- with output

$$\begin{aligned} y &= Cx + Du \\ &= \underbrace{CP^{-1}}z + \underbrace{D}u \end{aligned}$$

$$\begin{aligned} \tilde{A} &= PAP^{-1} \\ \tilde{B} &= PB \\ \tilde{C} &= CP^{-1} \\ \tilde{D} &= D \end{aligned}$$

Relating state-space models

- The equivalent state-space model in *z*

$$\dot{z} = \tilde{A}z + \tilde{B}u \qquad \tilde{A} = PAP^{-1}$$
$$y = \tilde{C}z + \tilde{D}u \qquad \tilde{B} = PB$$
$$\tilde{C} = CP^{-1}$$
$$\tilde{D} = D$$

$$z = Px$$
$$P^{-1}z = x$$

- has the transfer function

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\ &= CP^{-1}(sI - PAP^{-1})^{-1}PB + D \\ &= CP^{-1}(sPP^{-1} - PAP^{-1})^{-1}PB + D \\ &= CP^{-1}P(sI - A)P^{-1}PB + D \\ &= \boxed{C(sI - A)^{-1}B + D} \end{aligned}$$

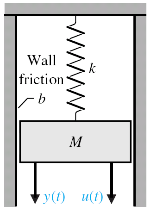
Example: Spring-Mass-Damper

- First-order differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t) \end{aligned}$$

- In matrix form:

$$\begin{aligned} \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$



- with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Example: Spring-Mass-Damper

- To find the transfer function
  - Plug the system matrices

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

  - into the formula

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example: Spring-Mass-Damper

Exercise:

- Find the new system matrices *A,B,C,D* that arise when the state is comprised of speed and position of the mass (in that order).
- Show that this system has the same transfer function as the system on the previous page.
- Bonus: What is the similarity transformation *P* that relates this system to the one on the previous page?

Summary

- Today and the last class
  - Linear algebra review 1
  - State-space models --> transfer functions

- Next time
  - Transfer function --> state-space models
  - Closed-form solution to state-space models
  - Linear algebra 2