EECE 360

Lecture 8

State Equation Representation of Dynamic Systems (cont'd)

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Chapter 3.3-3.5

Review: State-Space Model

State differential equation: $\dot{x}(t) = Ax(t) + Bu(t)$

Output equation: $y(t) = Cx(t) + Du(t)$

Taking the Laplace transform (assume zero initial conditions)

$$sX(s) = AX(s) + BU(s)$$

$$[sI - A]X(s) = BU(s)$$

$$\text{So,} \quad X(s) = [sI - A]^{-1}BU(s)$$

$$\boxed{\begin{aligned} Y(s) &= \{C[sI - A]^{-1}B + D\}U(s) \\ &= G(s)U(s) \end{aligned}}$$

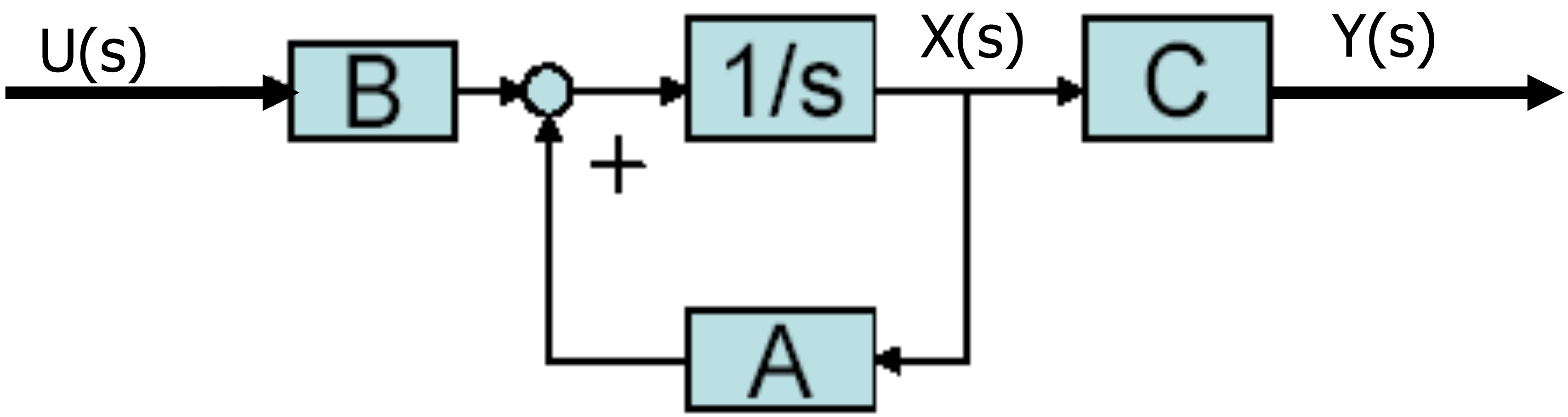
Using Matlab

The Matlab command **ss2tf** automates this process and returns vectors corresponding to the numerator and denominator of the system's transfer function.

```
>> help ss2tf
   To get started, select "MATLAB Help" from the Help menu.
  SS2TF  State-space to transfer function conversion.
    [NUM,DEN] = SS2TF(A,B,C,D,iu)  calculates the transfer function:
                NUM(s)          -1
        H(s) = -------- = C(sI-A) B + D
               DEN(s)
    of the system:
        .
        x = Ax + Bu
        y = Cx + Du
     from the iu'th input.  Vector DEN contains the coefficients of
    the denominator in descending powers of s.  The numerator
    coefficients are returned in matrix NUM with as many rows as there
    are outputs y.
```

State-Space Block Diagram

$$\text{With zero-initial state:} \quad sX(s) = AX(s) + BU(s)$$
$$\text{With D} = 0: \quad Y(s) = CX(s)$$



T.F. to State-Space

- For a given transfer function, there is no unique state space realization
- Engineering dictates the use of a realization of least order, a minimal realization
- A minimal realization is both **controllable** and **observable**.

Controllable and Observable

- A system is **completely controllable** if there exists a control $u(t)$ that can transfer any initial state $x(0)$ to any desired $x(t)$ in a finite time T.
- A system is **completely observable** if there exists a finite time T such that, given the input $u(t)$, the initial state $x(0)$ can be determined from the observation history $y(t)$.

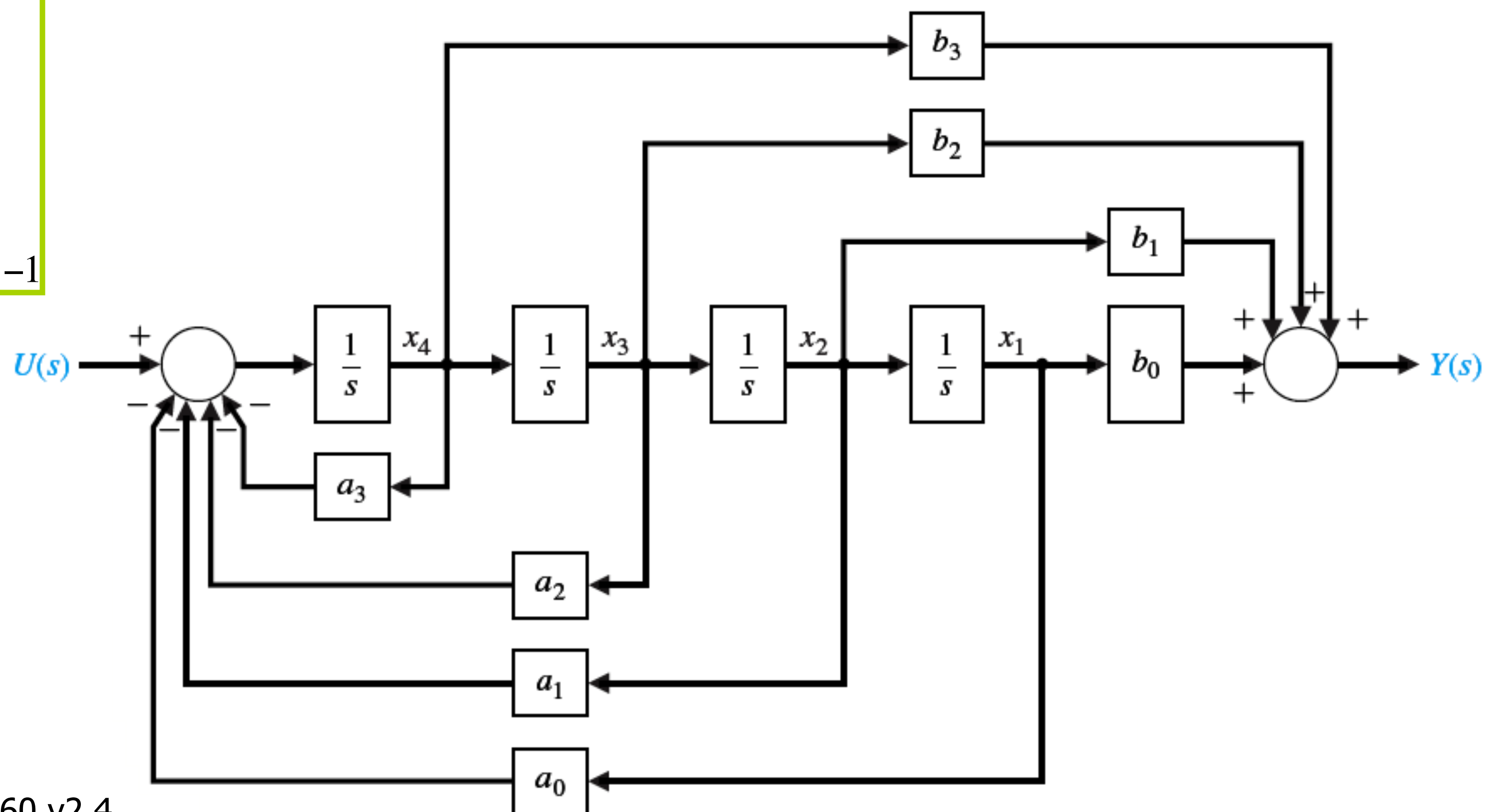
Canonical Forms

- The block diagram models can be easily derived from the transfer function from a system
- More than one alternative set of state variables
- More than one form of block diagram models
- Several key canonical forms of the state-space model
  - **Control canonical form**
  - **Observer canonical form**
  - Jordan canonical form
  - Diagonal canonical form
  - etc…

Control Canonical Form**

$$
\begin{aligned}
x_1 &= z \\
x_2 &= \dot{x}_1 \\
x_3 &= \dot{x}_2 \\
&\vdots \\
x_n &= \dot{x}_{n-1}
\end{aligned}
$$

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$



Control Canonical Form

- Transfer function is unchanged when

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} \frac{Z(s)}{Z(s)}$$

- Let

$$\begin{aligned} Y(s) &= \left(b_{n-1}s^{n-1} + \cdots + b_1 s + b_0\right) Z(s) \\ U(s) &= \left(s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0\right) Z(s) \end{aligned}$$

- Inverse Laplace yields high-order diff. equations in *z*

$$\begin{aligned} y(t) &= b_{n-1}z^{(n-1)} + \cdots + b_1 \dot{z} + b_0 z \\ u(t) &= z^{(n)} + a_{n-1}z^{(n-1)} + \cdots a_1 \dot{z} + a_0 z \end{aligned}$$

Control Canonical Form

- Choose the state vector

$$\boxed{\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 \\ &\vdots \\ x_n &= \dot{x}_{n-1} \end{aligned}}$$

- Substitute into

$$\begin{aligned} y(t) &= b_{n-1} z^{(n-1)} + \cdots + b_1 \dot{z} + b_0 z \\ u(t) &= z^{(n)} + a_{n-1} z^{(n-1)} + \cdots a_1 \dot{z} + a_0 z \end{aligned}$$

- To yield

$$\begin{aligned} y(t) &= b_{n-1} x_n + \cdots + b_1 x_2 + b_0 x_1 \\ u(t) &= \dot{x}_n + a_{n-1} x_n + \cdots a_1 x_2 + a_0 x_1 \end{aligned}$$

- Rearranging in state-space form,

$$\begin{aligned} \dot{x}_n &= -a_{n-1} x_n - \cdots - a_1 x_2 - a_0 x_1 + u \\ \dot{x}_{n-1} &= x_n, \;\; \dot{x}_{n-2} = x_{n-1}, \;\; \cdots, \;\; \dot{x}_1 = x_2 \end{aligned}$$

Control Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$ (Standard form)

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}, \quad D = 0$$

Observer Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Let $y(t) = x_1(t)$, then

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)X_n(s)$$

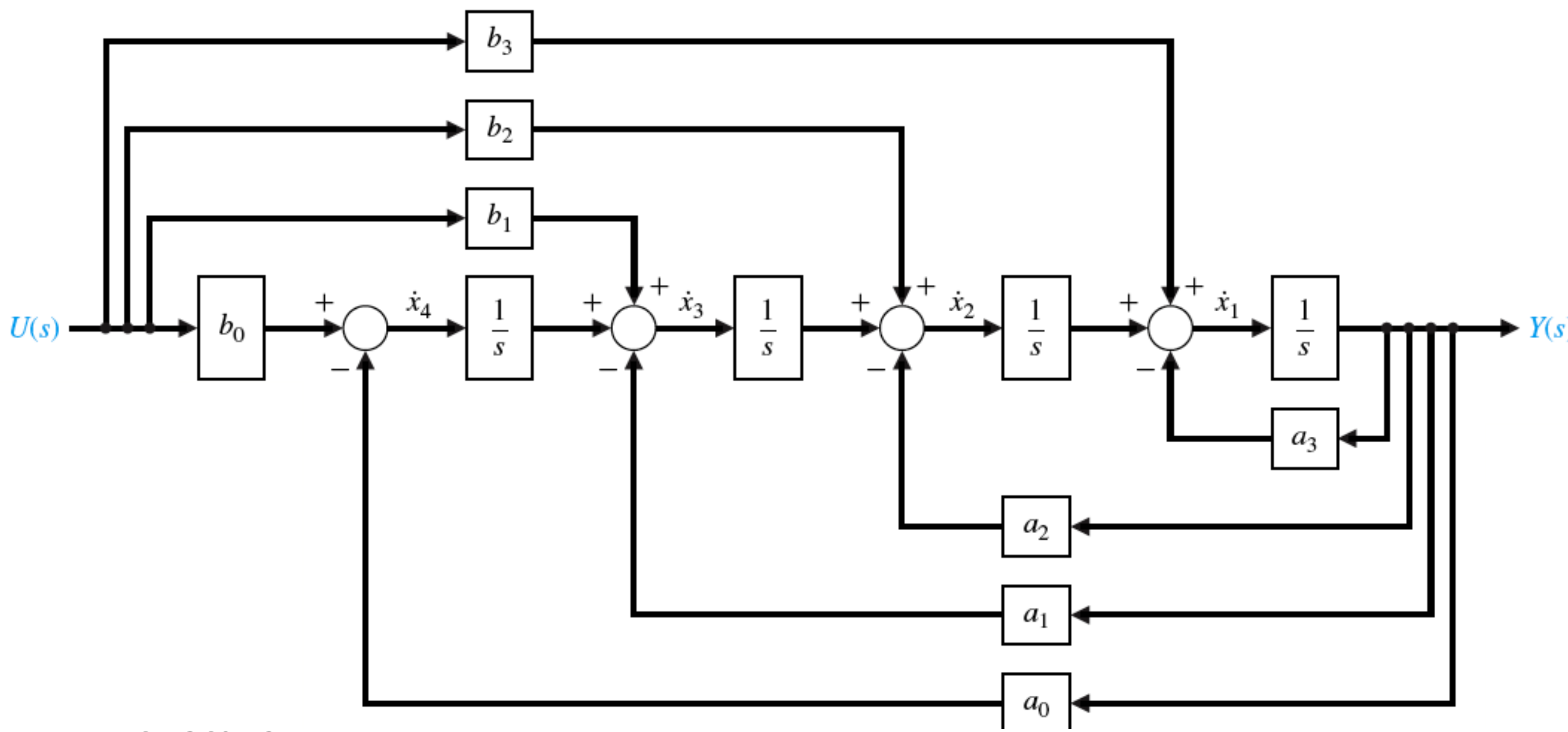
$$= (b_{n-1}s^{n-1} + \ldots + b_1 s + b_0)U(s)$$

$$\underbrace{-s^n X_n(s) + s^{n-1}(-a_{n-1}X_n(s) + b_{n-1}U(s)) + \ldots + \underbrace{s(-a_1 X_n(s) + b_1 U(s)) + \underbrace{(-a_0 X_n(s) + b_0 U(s))}_{sX_1(s)}}_{s^2X_2(s)}}_{s^nX_n(s)}$$

$$= 0$$

Observer Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$



Observer Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad D = 0$$

Example 1

- Given the transfer function

$$G(s) = \frac{s+2}{s^2 + 4s + 3}$$

- **1.** Find the state-space realization in control canonical form.
- **2.** Find the state-space realization in observer canonical form.

Example 1

$$G(s) = \frac{s+2}{s^2 + 4s + 3}$$

- **1. Control canonical form**

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 1 \end{bmatrix}, \qquad D = 0$$

- Results in the state-space realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

Example 1

$$G(s) = \frac{s+2}{s^2+4s+3}$$

- Confirm that this state-space realization produces the same transfer function:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 1

$$G(s) = \frac{s+2}{s^2+4s+3}$$

- and since $\left(\begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} \right)^{-1} = \frac{1}{s(s+4)+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$

- we substitute such that

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+4)+3} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s+3} \begin{bmatrix} 2(s+4)-3 & 2+s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+2}{s^2+4s+3}$$

Example 1

$$G(s) = \frac{s+2}{s^2+4s+3}$$

- **2. Observer canonical form**

$$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = 0$$

- Results in the state-space realization

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + 0 \cdot u$$

Example 1

$$G(s) = \frac{s+2}{s^2+4s+3}$$

- Confirm that this state-space realization produces the same transfer function:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 3 \\ -1 & s+4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example 1

$$G(s) = \frac{s+2}{s^2+4s+3}$$

- and since $\left(\begin{bmatrix} s & 3 \\ -1 & s+4 \end{bmatrix}\right)^{-1} = \frac{1}{s(s+4)+3}\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}$

- we substitute such that

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+4)+3}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{s+2}{s^2+4s+3}$$

Example 1

State-space to transfer function
Transfer function to state-space

- Control Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

Transfer function

$$G(s) = \frac{s+2}{s^2 + 4s + 3}$$

- Observer Canonical Form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + 0 \cdot u$$

The State Transition Matrix

- Previously, we found that

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\text{Thus,} \quad X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

- Now consider the homogenous (i.e. zero-input) case:

$$\dot{x} = Ax$$

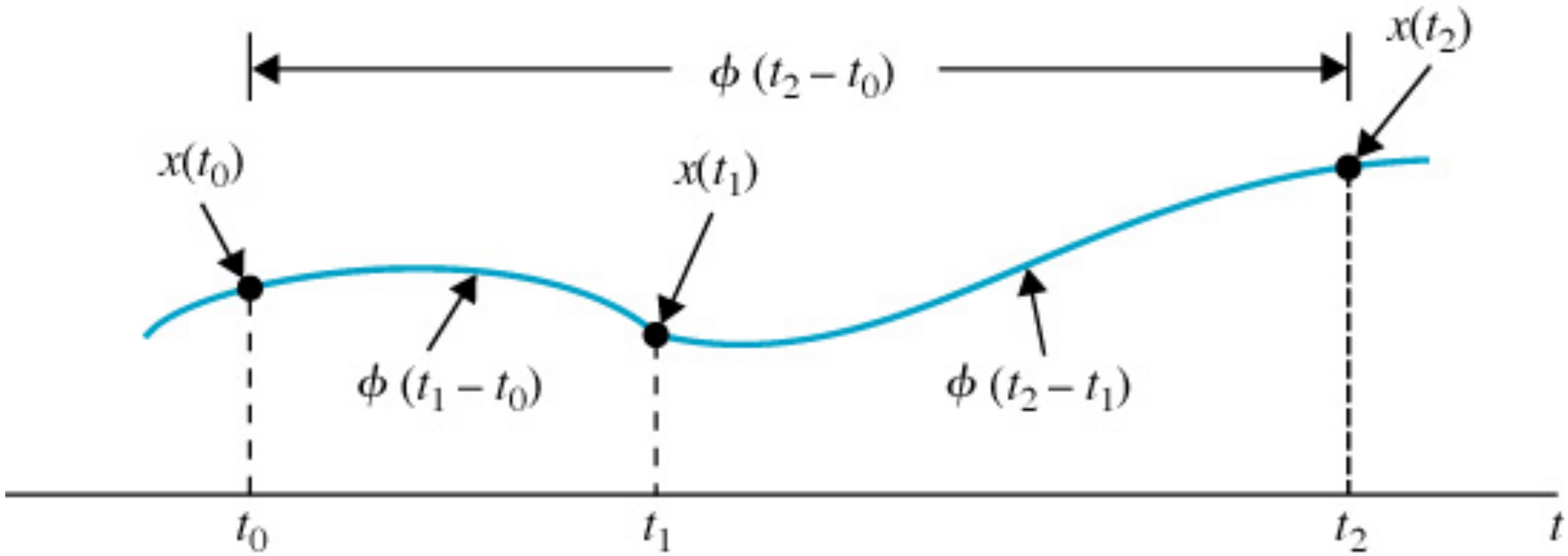
- The solution to this equation represents the evolution of the system's *free response* to non-zero initial conditions:

$$\boxed{x(t) = \Phi(t)x(0)}$$

State transition matrix

The State Transition Matrix

- Given an initial value, the state transition matrix predicts the state at any other time



- How do we compute this matrix?

State Transition Matrix

- Successive differentiation provides

$$\begin{aligned} \dot{x} &= Ax \\ \ddot{x} &= A\dot{x} = A^2 x \\ &\vdots \\ x^{(k)} &= A^k x \end{aligned}$$

- Evaluate at *t = 0*:

$$x^{(k)}(0) = A^k x(0)$$

The State Transition Matrix

- Taylor's expansion of *x(t)* around *t=0:*

$$
\begin{aligned}
x(t) &= x|_{t=0} + \frac{dx}{dt}|_{t=0}t + \frac{d^2x}{dt^2}|_{t=0}\frac{t^2}{2!} + \cdots + x^{(k)}|_{t=0}\frac{t^k}{k!} + \cdots \\
&= x(0) + \dot{x}(0)t + \ddot{x}(0)\frac{t}{2} + \cdots + x^{(k)}(0)\frac{t^k}{k!} + \cdots \\
&= x(0) + Ax(0)t + A^2x(0)\frac{t^2}{2!} + \cdots + A^kx(0)\frac{t^k}{k!} + \cdots \\
&= \left(I + At + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots\right)x(0)
\end{aligned}
$$

- This series converges for all time, and is known as the **matrix exponential function**.

$$
\boxed{x(t) \quad = \quad e^{At}x(0) = \Phi(t)x(0)}
$$

Example 2

- Consider the system

$$\dot{x} = Ax, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

with initial condition $x(0) = [1\ 1]^T$.

- What is the state at $t=1$? At $t=5$?

- Solution: Find

$$x(t) \quad = \quad e^{At}x(0) = \Phi(t)x(0)$$

Example 2

- Using the state transition matrix, we know that

$$x(1) = e^{A\cdot 1}x(0)$$

$$x(5) = e^{A\cdot 5}x(0)$$

- So we compute the matrix exponential

$$e^{At} = I + At + A^2\frac{t^k}{2!} + \ldots + A^k\frac{t^k}{k!} + \ldots = \sum_{k=1}^{\infty} A^k\frac{t^k}{k!}$$

- by finding

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \qquad A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -3^3 \end{bmatrix}$$

$$\vdots$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \qquad A^k = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-3)^k \end{bmatrix}$$

Example 2

- So we find the matrix exponential

$$e^{At} = I + At + A^2\frac{t^k}{2!} + \ldots + A^k\frac{t^k}{k!} + \ldots = \sum_{k=1}^{\infty} A^k\frac{t^k}{k!}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} t + \begin{bmatrix} (-1)^2 & 0 \\ 0 & (-3)^2 \end{bmatrix}\frac{t^2}{2!} + \ldots + \begin{bmatrix} (-1)^k & 0 \\ 0 & (-3)^k \end{bmatrix}\frac{t^k}{k!}$$

$$= \begin{bmatrix} \left(1 - t + \frac{t^2}{2!} + \ldots + (-1)^k\frac{t^k}{k!} + \ldots\right) & 0 \\ 0 & \left(1 - 3t + 3^2\frac{t^2}{2!} + \ldots + (-3)^k\frac{t^k}{k!} + \ldots\right) \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

- Now recall that for scalar exponentials

$$e^{\alpha t} = 1 + \alpha t + \alpha^2 \frac{t^k}{2!} + \ldots + \alpha^k \frac{t^k}{k!} + \ldots = \sum_{k=1}^{\infty} \alpha^k \frac{t^k}{k!}$$

- So for the above matrix A, the matrix exponential is

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

- The solution to $\dot{x} = Ax$ is

$$x(t) = e^{At} x(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix}$$

State Transition Matrix

- **Note that while for this example and for other diagonal matrices,

$$e^{At} = \begin{bmatrix} e^{a_{11}t} & 0 & \dots & 0 \\ 0 & e^{a_{22}t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_{22}t} \end{bmatrix}$$

- **But for general *A* the matrix exponential is NOT the exponential of each of its elements.**

State Transition Matrix

- The matrix exponential can be easily solved for some forms of *A* (diagonal, upper triangular, and others)
- **But for general *A*, an easier way to solve for the state transition matrix is to find its **Laplace transform $(sI-A)^{-1}$**.
- Can be computed in Matlab using '**expm**' for specific *A* and *t*

```
>>A=[0 -2; 1 -3]; dt=0.2; Phi=expm(A*dt)

Phi =

   0.9671   -0.2968
   0.1484    0.5219
```

State transition matrix for a Δt of 0.2 second

Summary

- Canonical forms
- State transition matrix
- State transition equation