EECE 360

Lecture 10

State Equation Representation of Dynamic Systems (cont'd)

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Chapter 3.7, Appendix E

Last class

- Canonical forms
  - Control canonical
  - Observer canonical
- State transition matrix $\Phi(t) = L^{-1}(\Phi(s))$
- Matrix exponential e^{At}
- Solution to $\dot{x}(t) = Ax(t) + Bu(t)$

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{Forced response}}$$

Review: State Trans. Matrix

- For the homogeneous system $\dot{x} = Ax$ we examined two ways to solve for *x(t):*
  - Time domain

    $$x(t) = \Phi(t)x(0), \qquad \Phi(t) = e^{At}$$

    $$\Phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + \ldots + \frac{t^k}{k!}A^k + \ldots$$
  - Laplace domain

    $$x(t) = L^{-1}(\Phi(s))x(0), \quad \Phi(s) = (sI - A)^{-1}$$
- It is often easier to solve for the state transition matrix in the Laplace domain

Review: State Trans. Matrix

- Solve $\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$ for x(t) in either the Laplace- or time-domain

$$X(s) = \underbrace{\Phi(s)x(0)}_{} + \underbrace{\Phi(s)BU(s)}_{}$$

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{Forced response}}$$

- Often easier to use the Laplace domain, then take the inverse Laplace transform of the result.

Review: State Trans. Matrix

- Why is this so useful?
- We now know how to solve *any* system that can be put into the state-space form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

for *any initial condition* $x(0)=x_0$ and *any input* $u(t)$.

- The solution is the sum of the **natural response** (zero-input case) and the **forced response** (zero-state case).

Today

- Characteristic equation
  - Relationship between state-space and transfer function forms
- Linear Algebra Review 2
  - Eigenvalues
  - Eigenvectors
- Putting it all together
  - Relating state-space, transfer function, and n^{th}-order differential equations representations
  - Example

Characteristic equation**

- For a transfer function *G(s)=N(s)/D(s)*
  - The **characteristic equation** is *D(s)=0*
  - The roots of the characteristic equation are the ***poles of G(s).***
- Recall that the denominator of the transfer function of a state-space representation is *det(sI-A)*
  - The **characteristic equation** is *det(sI-A)=0*
  - The roots of the characteristic equation are the **eigenvalues of the matrix *A*.**
- **The poles of G(s) are equal to the eigenvalues of A** (assuming no co-located poles and zeros).

Linear Algebra Review 2

- The roots of the characteristic equation, previously described as the poles of the transfer function *G(s),* are equivalent to the eigenvalues of the state matrix *A*.
- These values are important for analyzing system behavior and for designing good control laws.
- We need to know how to find the eigenvalues (and eigenvectors) of such a matrix.

Linear Algebra Review 2

- Eigenvalues and Eigenvectors
  - A nonzero vector v_i which satisfies
  $$Av_i = \lambda_i v_i,$$
  where λ_i is an eigenvalue of *A*, is the *eigenvector* associated with eigenvalue λ_i.
  - These are particular vectors for which the matrix *A* changes their magnitude, but not their direction.
  - If *A* has distinct eigenvalues, the eigenvectors can be found directly.

Linear Algebra Review 2

- Useful eigenvalue facts
  - If the coefficients of *A* are real, then the eigenvalues of *A* are either real, or complex conjugate pairs
  - The trace of *A* is the sum of all eigenvalues
  - Eigenvalues of *A* are also eigenvalues of A^T
  - If *A* is nonsingular, with eigenvalues λ_i^{-1}, then the eigenvalues of A^{-1} are λ_i

Linear Algebra Review 2

- Calculating eigenvalues:
  - Eigenvalues and eigenvectors must fulfill
  $$Av_i = \lambda_i v_i$$
  $$0 = (\lambda_i I - A)v_i$$
  - For non-zero v_i, the matrix $(\lambda_i I - A)$ must be singular. Therefore
  $$0 = \det(\lambda_i I - A)$$
  - and the *eigenvalues* of *A* are scalar values for which this holds.

Linear Algebra Review 2

- Calculating eigenvectors:
  - For a given eigenvalue, the corresponding eigenvector fulfills
  $$Av_i = \lambda_i v_i$$
  $$0 = (\lambda_i I - A)v_i$$
  - For each λ_i, find the matrix $(\lambda_i I - A)$. Pick the elements of v_i such that the above equation holds, and not all elements of v_i are zero.
- In Matlab, `[V,D]=eig(A)`
  - Columns of V are eigenvectors
  - Diagonal elements of D are eigenvalues

Example 1

- Problem
  - Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

- Solution
  - Eigenvalues

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 1 & 1 \\ 0 & \lambda + 1 \end{bmatrix}\right) = (\lambda - 1)(\lambda + 1)$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Example 1

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

- Solution
  - Eigenvectors
  - Case 1: $\lambda_1 = 1$

$$0 = (\lambda_1 I - A)v_1$$
$$= \begin{bmatrix} \lambda_1 - 1 & 1 \\ 0 & \lambda_1 + 1 \end{bmatrix}\begin{bmatrix} v_{1,a} \\ v_{1,b} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_{1,a} \\ v_{1,b} \end{bmatrix} = \begin{bmatrix} v_{1,b} \\ 2v_{1,b} \end{bmatrix}$$

  - Therefore one solution is $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example 1

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

- Solution
  - Eigenvectors
  - Case 2: $\lambda_2 = -1$

$$0 = (\lambda_2 I - A)v_2$$
$$= \begin{bmatrix} \lambda_2 - 1 & 1 \\ 0 & \lambda_2 + 1 \end{bmatrix}\begin{bmatrix} v_{2,a} \\ v_{2,b} \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{2,a} \\ v_{2,b} \end{bmatrix} = \begin{bmatrix} -2v_{2,a} + v_{2,b} \\ 0 \end{bmatrix}$$

  - Therefore one solution is $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 1

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

- Solve in Matlab

```
>> A = [1 -1; 0 -1]
A =
     1    -1
     0    -1

>> [V,D]=eig(A)
V =
    1.0000     0.4472
         0     0.8944
D =
     1      0
     0     -1
```

- Check results

```
>> (D(2,2)*eye(2,2)-A)*V
   (:,2)

ans =
     0
     0
```

Linear Algebra Review 2

- Calculating eigenvectors
  - Eigenvectors must be linearly independent (e.g. cannot be a linear combination of other eigenvectors)

$$v_i \neq \sum_{j \neq i} \alpha_j v_j$$

  - Repeated eigenvalues require additional work to find independent eigenvectors (multiply by t, t^2, etc.)
  - Eigenvectors must be non-zero (e.g. cannot have all elements of any eigenvector equal to zero)

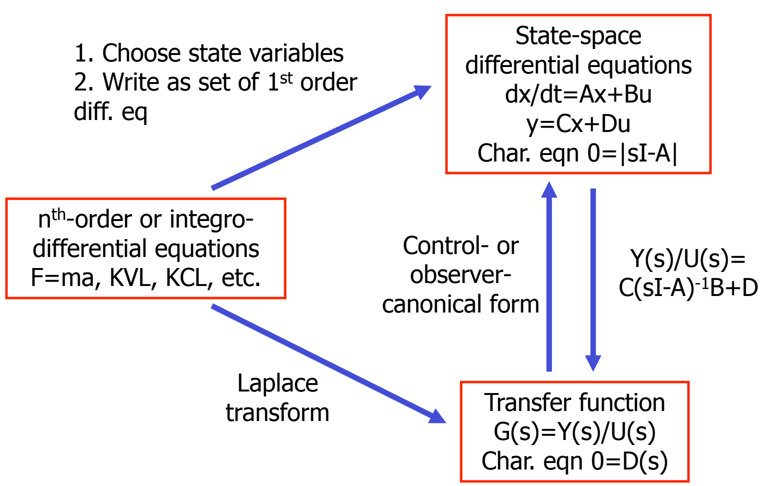
Linear Algebra Review 2

- Eigenvalues of *A*:
  - Find λ_i such that $0 = \det(\lambda_i I - A)$
- Eigenvectors of *A*:
  - Find v_i such that

$$Av_i = \lambda_i v_i$$
$$0 = (\lambda_i I - A)v_i$$

- Computing eigenvalues and eigenvectors in Matlab, `[V,D]=eig(A)`

Putting it all together



Putting it all together

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$
$$\Phi(t) = e^{At}$$

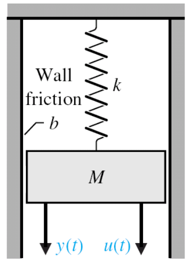
Inverse Laplace transform ↑ ↓ Laplace transform

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$
$$\Phi(s) = (sI - A)^{-1}$$

Example: Spring-Mass-Damper

Problem:

- Consider the spring-mass damper system with input u(t) and output y(t), the position of the mass. With state $x = [x_M, v_M]^T$:
  - What is the *state response* to an initial condition $x_0=[1\ 0]^T$ and an impulse input?
  - What is the *output response*? Assume k=1, b=1, M=1.



Solution:

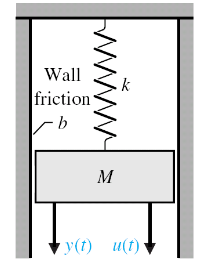
- Identify equations of motion (F=ma)
- Find state-space description with states *x*, *v*
- Solve for the state transition matrix
- Find x(t) and y(t) as functions of time only.

Example: Spring-Mass-Damper

- Recall the state is $x = [x_M, v_M]^T$
- Find dx_M/dt and dv_M/dt in terms of x_M, v_M, and u

$$\dot{x}_M(t) = v_M(t)$$

$$\dot{v}_M(t) = -\frac{k}{M}x_M(t) - \frac{b}{M}v_M(t) + \frac{1}{M}u(t)$$



- In matrix form:

$$\begin{bmatrix}\dot{x}_M(t)\\ \dot{v}_M(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix}\begin{bmatrix}x_M(t)\\ v_M(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_M(t)\\ v_M(t)\end{bmatrix} + 0\cdot u(t)$$

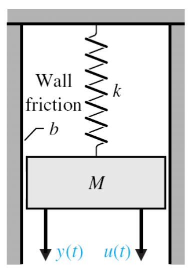
Example: Spring-Mass-Damper

- Find the state transition matrix

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix}s & -1\\ \frac{k}{M} & s+\frac{b}{M}\end{bmatrix}^{-1}$$

$$= \frac{1}{s^2 + \frac{b}{M}s + \frac{k}{M}}\begin{bmatrix}s+\frac{b}{M} & 1\\ -\frac{k}{M} & s\end{bmatrix}$$

$$= \frac{1}{s^2+s+1}\begin{bmatrix}s+1 & 1\\ -1 & s\end{bmatrix}$$



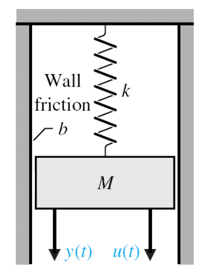
- Plug into the formula $X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$

$$\Phi(s)x(0) = \frac{1}{s^2+s+1}\begin{bmatrix}s+1 & 1\\ -1 & s\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{s^2+s+1}\begin{bmatrix}s+1\\ -1\end{bmatrix}$$

Example: Spring-Mass-Damper

$$\Phi(s)BU(s) = \frac{1}{s^2+s+1}\begin{bmatrix}s+1 & 1\\ -1 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\cdot 1$$

$$= \frac{1}{s^2+s+1}\begin{bmatrix}1\\ s\end{bmatrix}$$



- Therefore the cumulative response is

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$

$$= \frac{1}{s^2+s+1}\left(\begin{bmatrix}s+1\\ -1\end{bmatrix} + \begin{bmatrix}1\\ s\end{bmatrix}\right) = \frac{1}{s^2+s+1}\begin{bmatrix}s+2\\ s-1\end{bmatrix}$$

$$x(t) = L^{-1}\left(\frac{1}{s^2+s+1}\begin{bmatrix}s+2\\ s-1\end{bmatrix}\right)$$

Example: Spring-Mass-Damper

- In order to find $x(t) = L^{-1}\left(\frac{1}{s^2+s+1}\begin{bmatrix} s+2 \\ s-1 \end{bmatrix}\right)$
- Rewrite the denominator as

$$s^2+s+1=(s+a)^2+\omega^2,$$
$$a=\tfrac{1}{2}, \omega=\tfrac{\sqrt{3}}{2}$$

Laplace transform pairs

$$\frac{s+a}{(s+a)^2+\omega^2} \Leftrightarrow e^{-at}\cos(\omega t)$$
$$\frac{\omega}{(s+a)^2+\omega^2} \Leftrightarrow e^{-at}\sin(\omega t)$$

- Therefore the first element of X(s) is

$$\frac{s+2}{s^2+s+1}=\left(\frac{s+\frac{1}{2}}{s^2+s+1}\right)+\sqrt{3}\left(\frac{\frac{\sqrt{3}}{2}}{s^2+s+1}\right)$$
$$x_M(t)=e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right)+\sqrt{3}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$

Example: Spring-Mass-Damper

- Similarly, the second element of *X(s)* is

$$\frac{s-1}{s^2+s+1}=\left(\frac{s+\frac{1}{2}}{s^2+s+1}\right)-\sqrt{3}\left(\frac{\frac{\sqrt{3}}{2}}{s^2+s+1}\right)$$
$$v_M(t)=e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right)-\sqrt{3}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$

- Answer to Problem 1: The *state response* is

$$x_M(t)=e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right)+\sqrt{3}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$
$$v_M(t)=e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right)-\sqrt{3}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$

Example: Spring-Mass-Damper

- Answer to Problem 2: The *output response* is

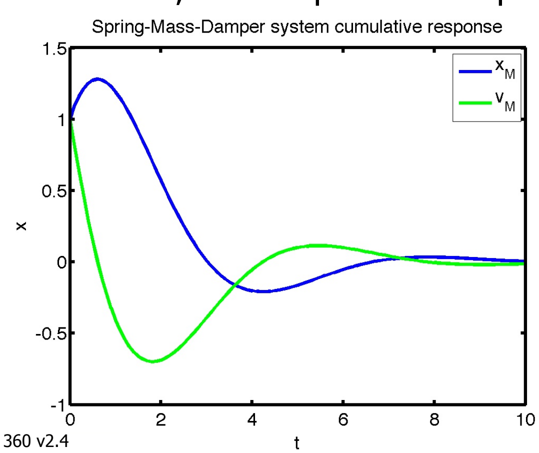
$$\begin{aligned} y(t) &= Cx(t)+Du(t) \\ &= \begin{bmatrix}1 & 0\end{bmatrix}x(t)+0\cdot u(t) \\ &= x_M(t) \end{aligned}$$

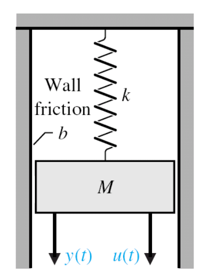
$$y(t)=e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right)+\sqrt{3}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right)$$

- *Question*: What is another way to find the output response for a given input and initial condition?

Example: Spring-Mass-Damper

- In Matlab, we compute the response as





Summary

Today

- Solution to the general state-space equations
- Characteristic equation
  - From state-space description
  - From transfer function
  - Finding eigenvalues and eigenvectors
- Putting it all together

Next class

- Feedback characteristics
  - Sensitivity, complementary sensitivity