EECE 360
Lecture 13

Feedback Characteristics

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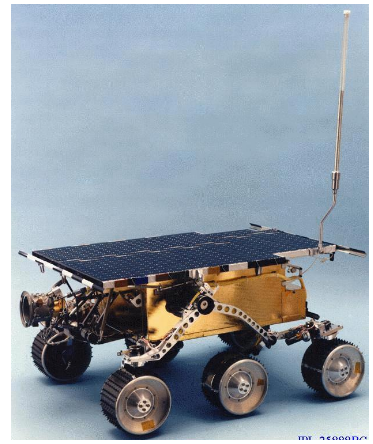
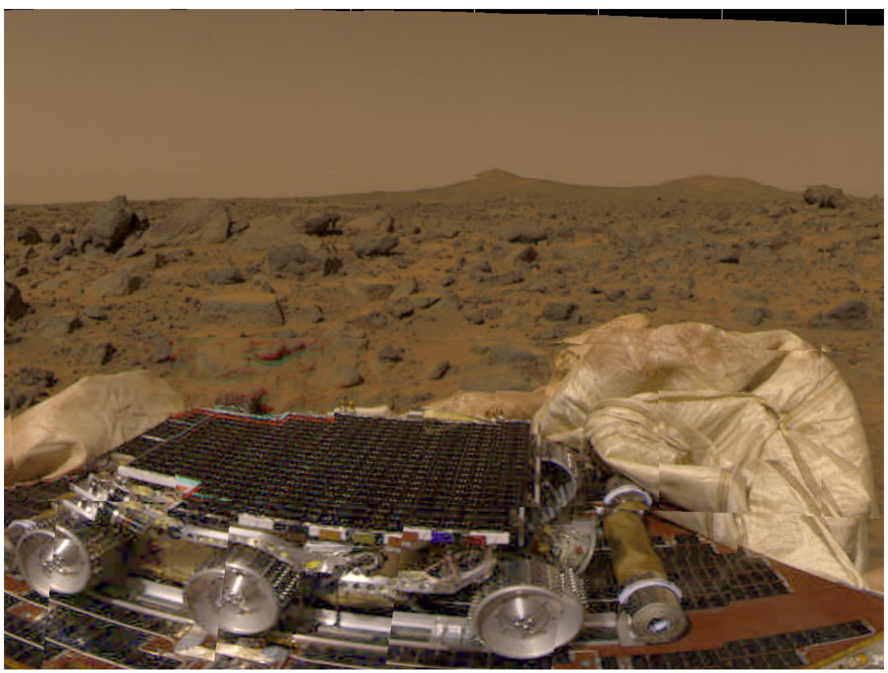
Chapter 4.8, 5.2-5.4

Example 1: Mars Rover

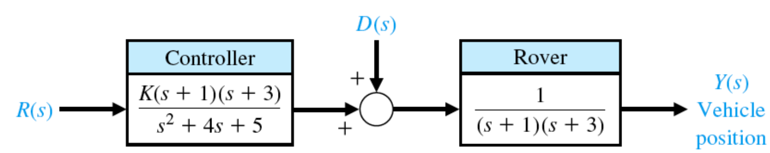


- Solar-powered *Sojourner*
- Launched in December 1996
- Landed July 4, 1997
- Remotely operated from earth



Example 1: Mars Rover

- Goal: Operate the rover with modest effects from external disturbances and with low sensitivity to the change in the gain *K*.
- Open-loop configuration

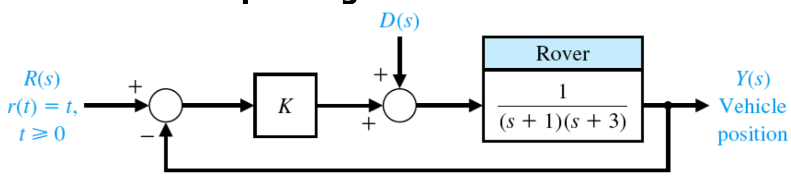


$$Y(s) = G(s)(D(s) + K(s)R(s))$$

$$= \frac{1}{(s+1)(s+3)} D(s) + \frac{K}{s^2 + 4s + 5} R(s)$$

Example 1: Mars Rover

- **Closed-loop configuration**

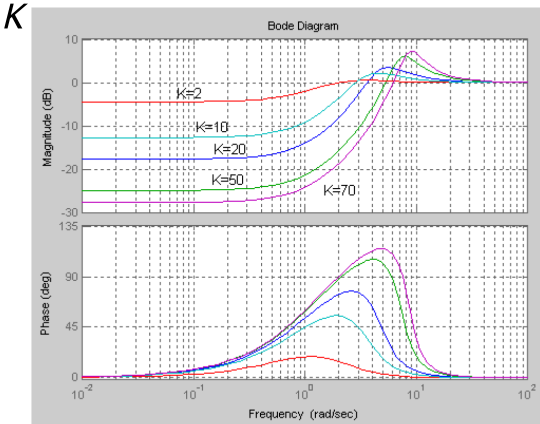


$$
\begin{aligned}
Y(s) &= G(s)(D(s) + K(s)(R(s) - Y(s))) \\
&= \frac{G(s)}{1 + G(s)K(s)} D(s) + \frac{G(s)K(s)}{1 + G(s)K(s)} R(s) \\
&= \frac{1}{(s+1)(s+3)+K} D(s) + \frac{K}{(s+1)(s+3)+K} R(s)
\end{aligned}
$$

Example 1: Mars Rover

- Sensitivity function (effect of noise on output) for varying *K*

$$
\begin{aligned}
S &= \frac{G(s)}{1 + G(s)K(s)} \\
&= \frac{1}{(s+1)(s+3)+K} \\
&= \frac{1}{s^2 + 4s + (3+K)}
\end{aligned}
$$



Example 1: Mars Rover

- Steady-state behavior to *D(s)= 1/s, R(s)=0*
  - Open-loop

$$
\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{s}{(s+1)(s+3)} \cdot \frac{1}{s} = \frac{1}{3}
$$

  - Closed-loop

$$
\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{s}{(s+1)(s+3)+K} \cdot \frac{1}{s} = \frac{1}{3+K}
$$

Outline

- Today
  - Second-order systems
  - Time domain specifications
  - Test input signals
  - Similar systems (3rd order, 2nd order with zeros)
- Next class
  - Input type and system type number
  - Steady-state error

Second-order systems

- Common performance measures
  - Transient response
  - Steady-state response
- Common test input signals to evaluate system response
  - Impulse
  - Step
  - Ramp
  - (Parabola)
- System performance is determined by the location of the poles

Second-order systems

- Why do poles determine system response?
- Recall that for $\dot{x}(t) = Ax(t) + Bu(t)$ the total response is

$$x(t) = \boxed{\Phi(t)}x(0) + \int_0^t \boxed{\Phi(t-\tau)}Bu(\tau)d\tau$$

where $\Phi(t) = e^{At}$

- The poles are determined by

$$0 = \det(\Phi(s)) = \det(sI - A)^{-1}$$

which are the **eigenvalues of *A*.**

Second-order systems

- Generic second-order system

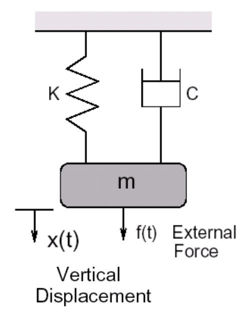
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - Natural frequency ω_n
  - Damping ratio ζ
- Characteristic equation

$$0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Spring-Mass-Damper System

- Input *u=f(t)*, Output *y = x(t)*



$$u(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

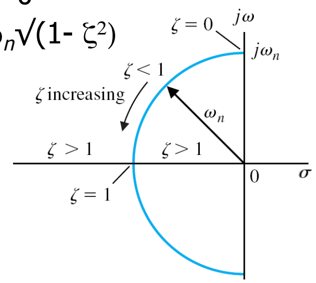
$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second-order systems

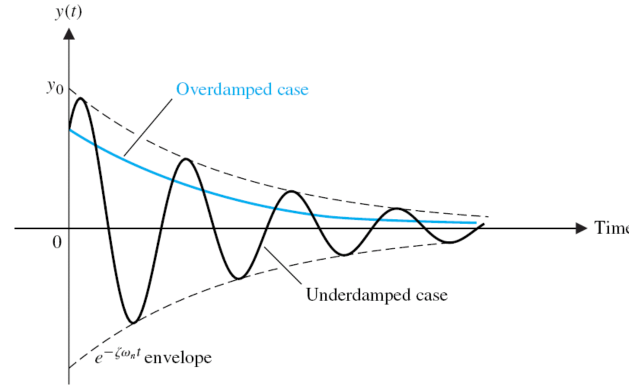
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Underdamped
  - Natural frequency $\omega_n > 0$
  - Damping ratio $1 > \zeta > 0$
  - **Damped frequency** $\omega_d = \omega_n\sqrt{(1-\zeta^2)}$
- Critically damped
  - Damping ratio $\zeta = 1$
- Overdamped
  - Damping ratio $\zeta > 1$



Natural Response (no input)

$$\frac{w_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Leftrightarrow \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t), \zeta < 1$$



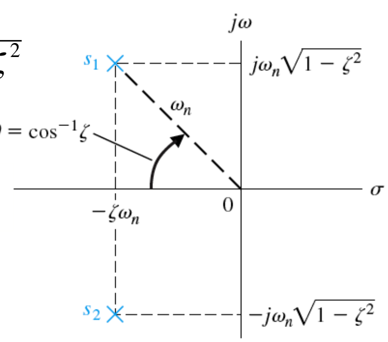
Second Order System Poles

- Underdamped
  - Complex roots due to the quadratic

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\sigma \pm j\omega_d$$

for $\zeta < 1$ and $\sigma = \zeta\omega_n$

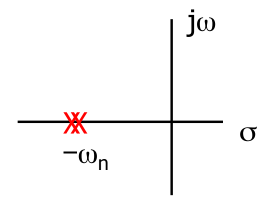


Second Order System Poles

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

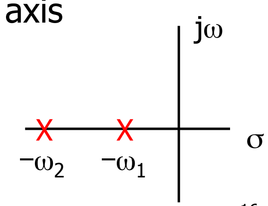
- Critically damped
  - Repeated poles

$$s_{1,2} = -\omega_n$$
since $\zeta = 1$

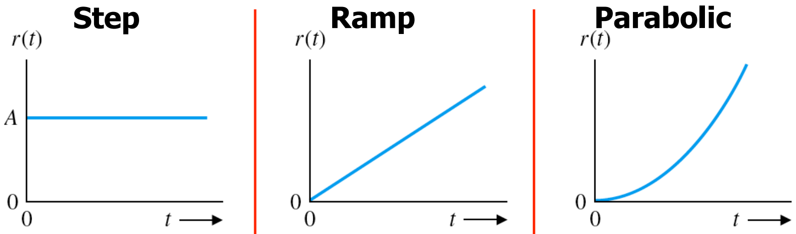


- Overdamped
  - Unique poles on the real axis

$$s_{1,2} = -\omega_1, -\omega_2$$
since $\zeta > 1$



Test Input Signals



$$r(t) = \begin{cases} A & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad R(s) = \frac{A}{s}$$

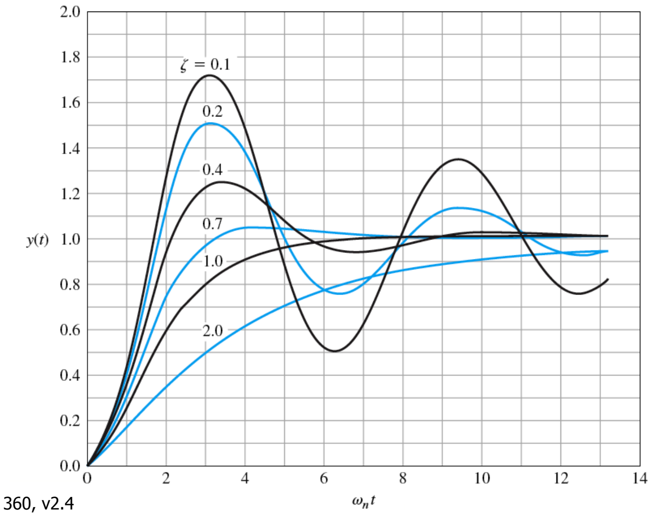
$$r(t) = \begin{cases} At & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad R(s) = \frac{A}{s^2}$$

$$r(t) = \begin{cases} At^2 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad R(s) = \frac{2A}{s^3}$$

- "Base-case" used to evaluate system response.

Second-Order System

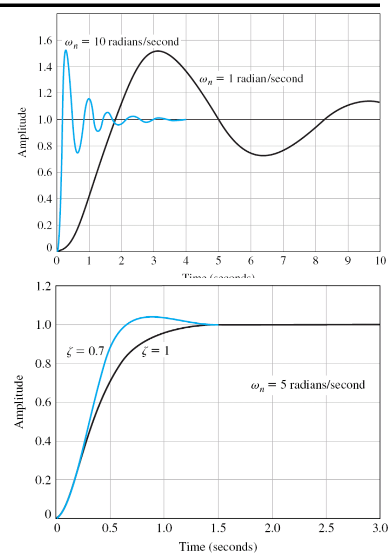
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



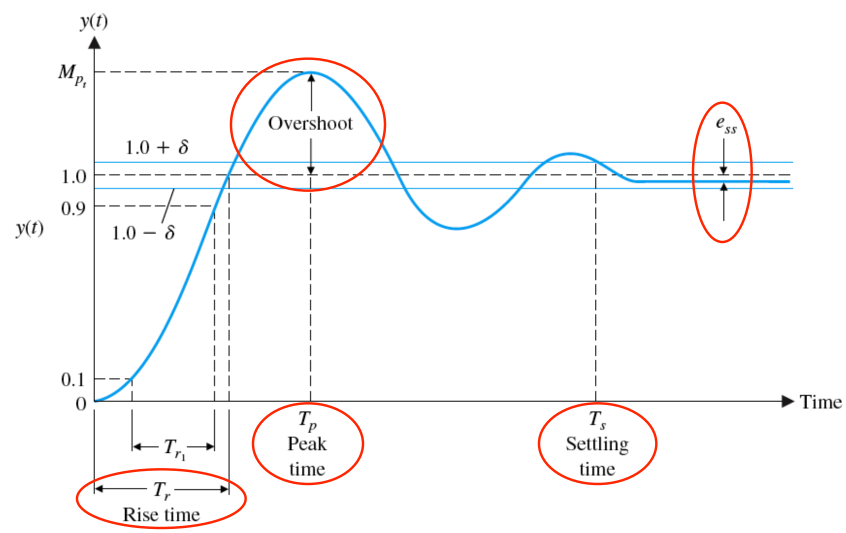
Second-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Effect of ω_n
  - Frequency of oscillations
- Effect of ζ
  - Damping



Second-Order System



Second-Order System

- Rise time
  - Time it takes output to reach the vicinity of its new set point.

$$T_r = \frac{2.16\zeta + 0.60}{\omega_n}$$

- Peak time
  - Time it takes the output to reach the maximum overshoot point

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- Overshoot
  - Maximum amount the output overshoots its final value, divided by its final value (usually a %age)

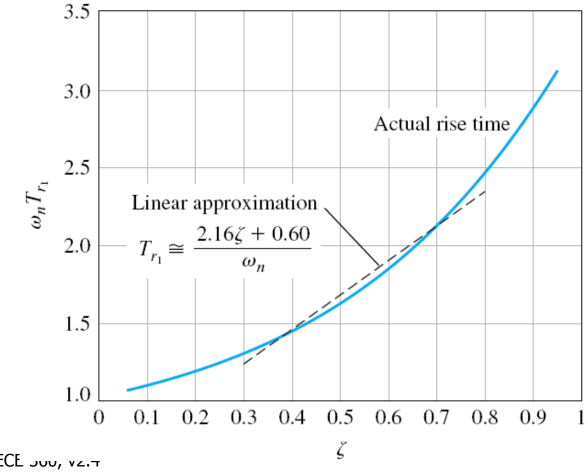
$$M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

- Settling time
  - Time it takes the transients to decay to 2% of final value

$$T_s = \frac{4}{\zeta\omega_n}$$

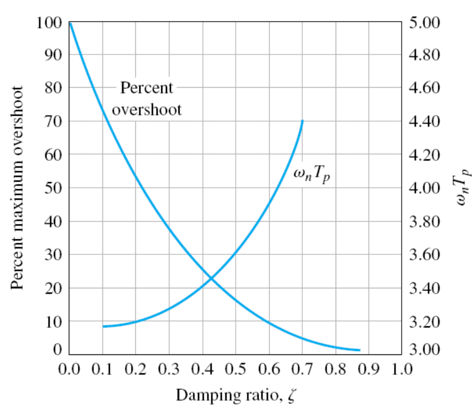
Second-Order System

Normalized rise time



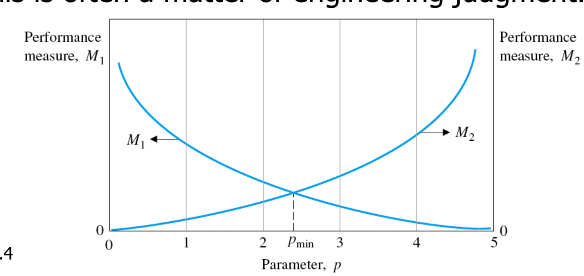
Second-Order System

- Trade-off between specifications



When specifications conflict

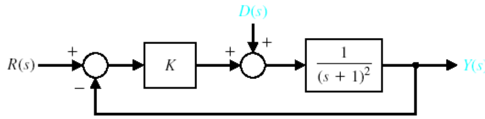
- Design specifications may be conflicting
  - For example, several time domain specifications as well as a steady-state error specification
- It *may not be possible* to meet all specifications.
- Find a compromise which is the “best” solution.
- This is often a matter of engineering judgment.

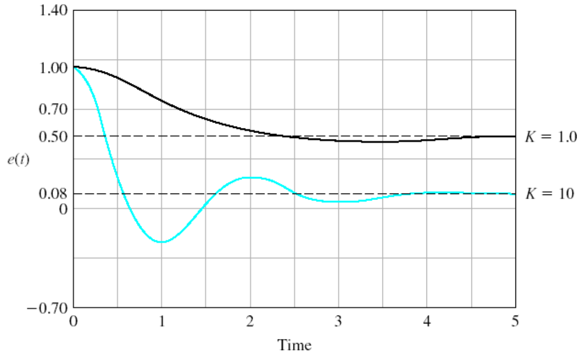


Example 2: A Simple Loop

The benefit of feedback can be illustrated:

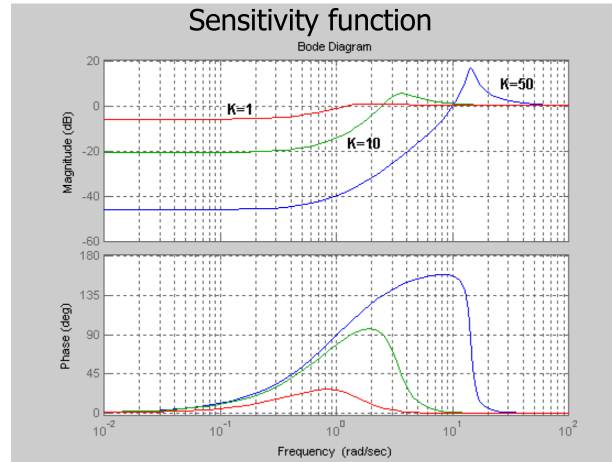
The rise time and sensitivity of the system are reduced as K increases.





For a unit step disturbance

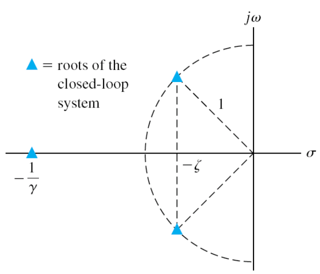
Example 2: A Simple Loop



Effect of a Third Pole

- For the third-order system

$$G(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s\gamma + 1)}$$



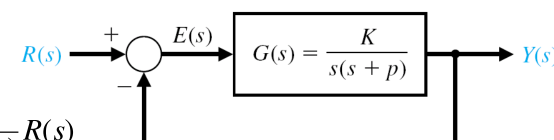
Experiments show that when

$$\left|\frac{1}{\gamma}\right| \geq 10\left|\zeta\omega_n\right|$$

the third-order system can be approximated by a second-order system to meet specifications for **overshoot** and **settling time**

Example 2

- Find K and p such that the overshoot is less than 5% and settling time less than 4 seconds



$$Y(s) = \frac{G(s)}{1+G(s)}R(s)$$

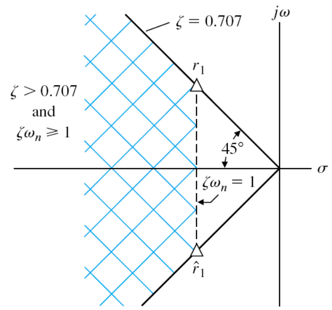
$$= \frac{K}{s^2 + ps + K}R(s)$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}R(s)$$

Example 2

- Damping ratio ζ = 0.707 provides an overshoot of 4%.
- Settling time is determined by

$$T_s = \frac{4}{\zeta\omega_n} < 4$$
$$\therefore \zeta\omega_n > 1$$



Example 2

- With $\zeta = 1/\sqrt{2}$, $\omega_n = \sqrt{2}$ the poles will be located at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\tfrac{1}{\sqrt{2}}\cdot\sqrt{2} \pm j\sqrt{2}\cdot\sqrt{1-\left(\tfrac{1}{\sqrt{2}}\right)^2}$$
$$= -1 \pm j$$

- We can find *K* and *p* by matching the coefficients of the characteristic equation

$$s^2 + ps + K = s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}$$
$$p = 2\zeta\omega_n = 2$$
$$K = \omega_n^{\ 2} = 2$$

Summary

- Today
  - Test input signals
  - Second-order systems
  - Performance characteristics
- Next class
  - Steady-state error
  - Type number