EECE 360
Lecture 15

Stability

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Chapter 6.1 - 6.3

Context

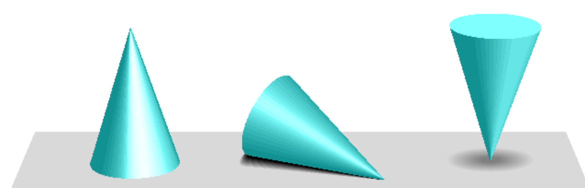
- Modeling LTI systems
  - S-domain
  - State-space
- Feedback characteristics
  - Transient response
  - Steady-state response
- Stability
- Next week: Techniques to analyze and design controllers for stability and other feedback characteristics

The Concept of Stability

- Two different notions of stability
  - Ability of the system to return to equilibrium after an arbitrary displacement away from the equilibrium
  - Ability of the system to produce a bounded output for any bounded input
- For linear systems, these two notions are closely related.

The Concept of Stability

- Perturbations from equilibrium



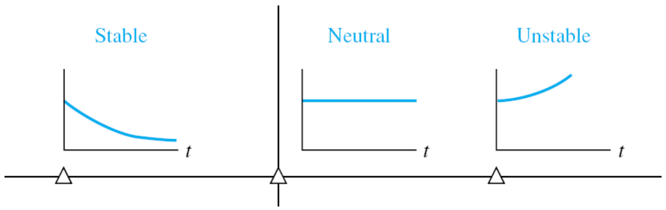
(a) Stable (b) Neutral (c) Unstable

If tipped slightly, the cone returns to its original position

If released, the cone falls onto its side

The Concept of Stability

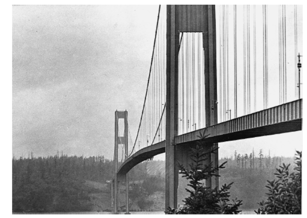
- Bounded input/bounded output



- The transient response is characterized by the location in the s-plane of the poles of a system
- Poles in the right-hand side of the s-plane result in an increasing response.

Tacoma Narrows, 1940

- A famous example of instability
- Amplitude of bridge oscillations grow until structural failure
- (Bounded input leads to unbounded output)



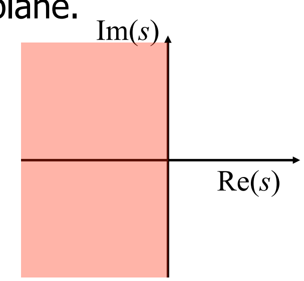
Types of Stability

- Internal Stability Stability *(perturbations from the equilibrium)*
  - Main focus of linear dynamical systems theory
- BIBO stability *(Bounded Input/Bounded Output)*
  - Main focus of this class
- Relative stability

Internal Stability**

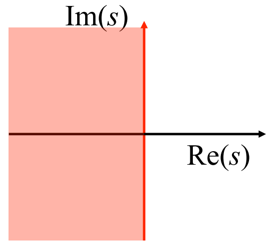
- A linear system is **internally stable** if and only if all roots of the characteristic equation have **negative real part.**
- This is equivalent to requiring that the poles of the system transfer function (=eigenvalues of A) lie in the open left half-plane.
- For an **internally stable** system, starting from **any** initial condition

$$x_0 \in \Re^n, \quad \boxed{x(t) \to 0 \text{ as } t \to \infty}$$

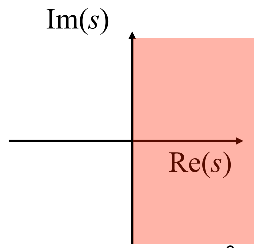


Internal Stability

- A linear system with non-repeated poles with zero real part and the remaining poles with negative real part is **stable in the sense of Lyapunov**.

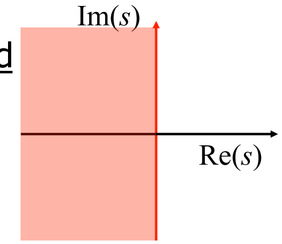


- An LTI system with at least one pole in the right half plane (RHP) is **unstable**.



Internal Stability

- A linear system with repeated poles (=co-located poles) on the imaginary axis, and all other poles in the LHP, is either **stable** or **unstable.**
- It cannot be asymptotically stable.
- Stability of these types of systems requires further analysis of eigenvectors (beyond the scope of this class).



Internal Stability

Examples: Find the stability of $\dot{x} = Ax$ for

- $A = I$
- $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$
- $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$
- $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

Exponential stability

- Internally stable LTI systems are also **asymptotically stable**

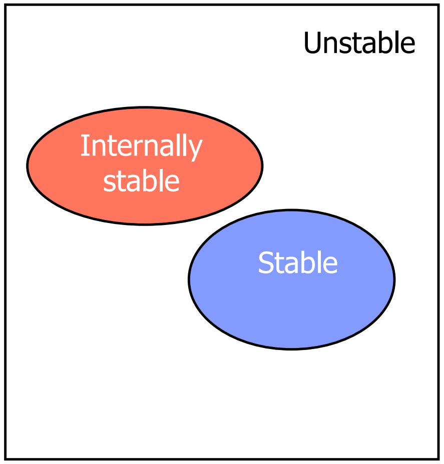
  (meaning that x(t) asymptotically approaches 0 as t approaches infinity)

- As well as **exponentially stable**

  (meaning that x(t) is bounded by an exponentially decaying function)

- Faster convergence of exponentially stable systems (as opposed to asymptotically stable systems) means better performance

Internal stability



BIBO Stability

- "Bounded-Input/Bounded-Output"
- A system is said to be BIBO stable if for *any* bounded input, the output remains bounded for all time.

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

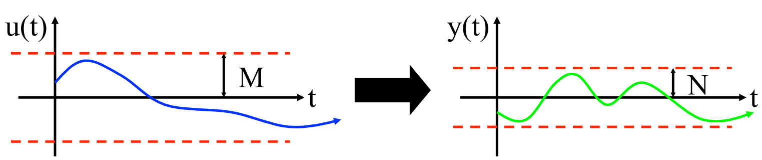
$$|y(t)| \le \int_0^t |g(t-\tau)| \cdot |u(\tau)| d\tau$$

$$\le M \int_0^t |g(t-\tau)| d\tau$$

- Internal stability ==> BIBO stability

BIBO Stability

- A bounded input is one for which $|u(t)| \le M$
- Which inputs are bounded?
  - $u(t) = \mathbf{1}(t)$
  - $u(t) = \delta(t)$
  - $u(t) = e^{at}$, $a < 0$ for $t \ge 0$
  - $u(t) = e^{at}$, $a = j\omega$

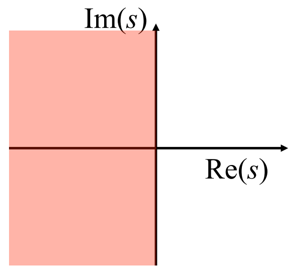


BIBO Stability

- Example:
  - Is the system with transfer function G(s) = 1/(s+1) BIBO stable?
- Answer:
  - Show that $y(t) \le M_y$ for all t and for some positive constant M_y.
  - Could alternatively show that G(s) is internally stable and therefore also BIBO stable
- **Can determine BIBO stability by simply examining poles of characteristic equation.**

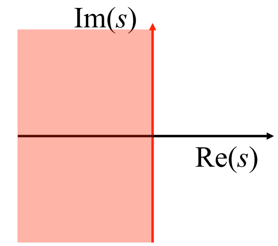
BIBO Stability**

- A linear system is **BIBO stable** if all poles of the transfer function have **negative real part.**
- For systems in which there is no pole-zero cancellation in the transfer function, this is the same as requiring all roots of the characteristic equation to be in the open LHP.



BIBO Stability

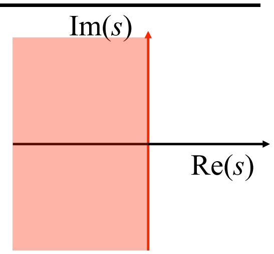
- A system with non-repeated poles with on the imaginary axis and the remaining poles with negative real part is **marginally stable**.

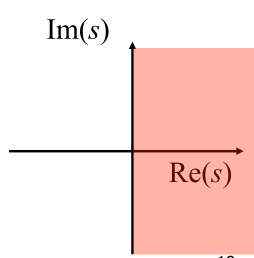


- Sinusoidal input at the frequency of the poles on the imaginary axis will cause unbounded output.

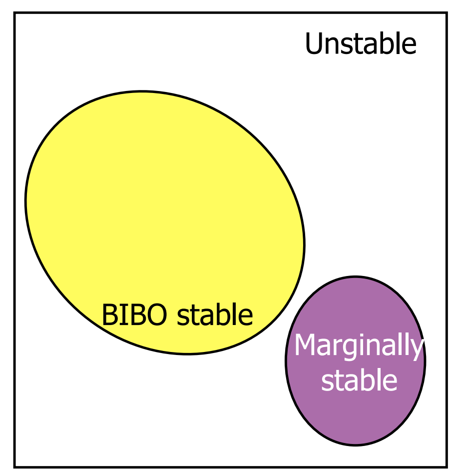
BIBO Stability

- An LTI system with at least one pole in the right half plane (RHP) is **unstable**.
- An LTI system with repeated poles on the imaginary axis is **unstable**.





BIBO stability



BIBO Stability**

- Assuming there is no pole-zero cancellation in the transfer function, and $\lambda_i,\ i \in \{1,\cdots,n\}$ are roots of the n^{th} order characteristic equation:

Stability	Pole location	Description
BIBO stable	A λ_i, $Re(\lambda_i)<0$	All poles in open LHP
Marginally stable	$\exists\lambda_i$, $Re(\lambda_i)=0$, $\lambda_i \neq \lambda_j \wedge \sim\exists\lambda_k$, $Re(\lambda_k)>0$	Any simple poles on imaginary axis, and no poles in RHP
Unstable	$\exists\lambda_i$, $Re(\lambda_i)=0$, $\lambda_i=\lambda_j$	Any repeated poles on imaginary axis, or
	$\exists\lambda_i$, $Re(\lambda_i)>0$	Any poles in RHP

Example

- Determine whether system with the characteristic equation

$$Q(s) = s^4 + 2s^3 + s^2$$

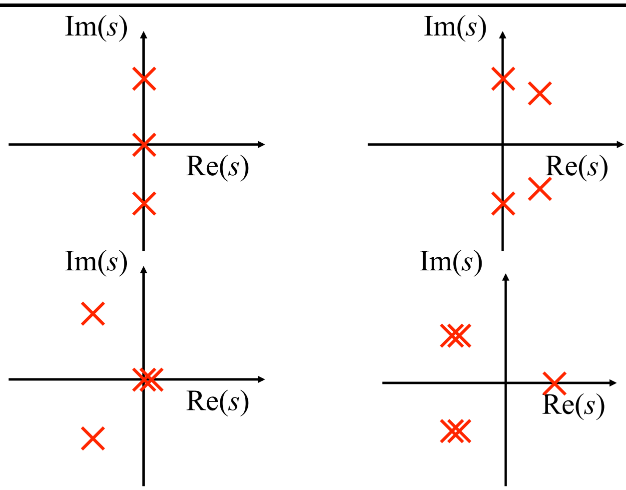
is BIBO stable.

- Since the polynomial

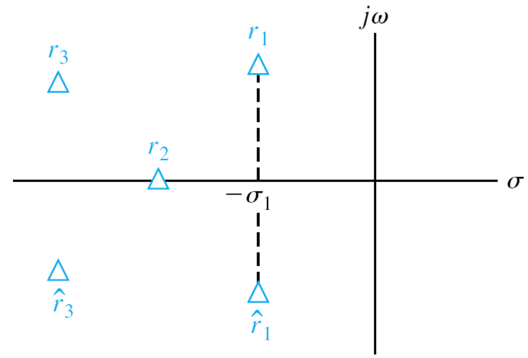
$$Q(s) = s^4 + 2s^3 + s^2 = s^2(s+1)^2$$

has repeated roots on the imaginary axis, it is **BIBO unstable**.

Examples



Relative Stability



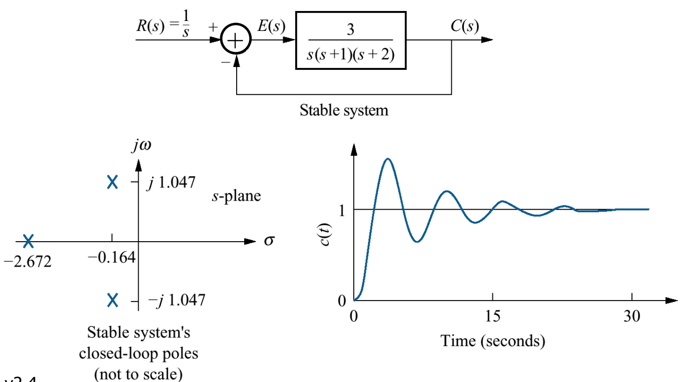
Relative stability: measured by the distance of the root from the real axis.

One algorithm to ascertain relative stability: the shifting of the s-plane axis, then check Routh-Hurwitz criteria.

Thus, root r_2 is relatively more stable than the root r_1.

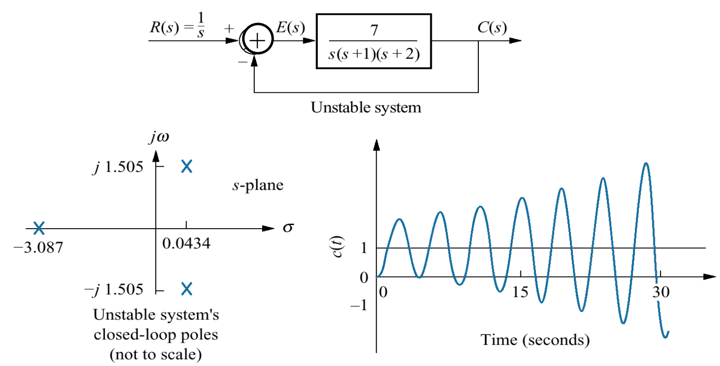
Feedback and Stability

- Consider the unity feedback system with gain K=3



Feedback and Stability

- Consider the same system with gain K = 7

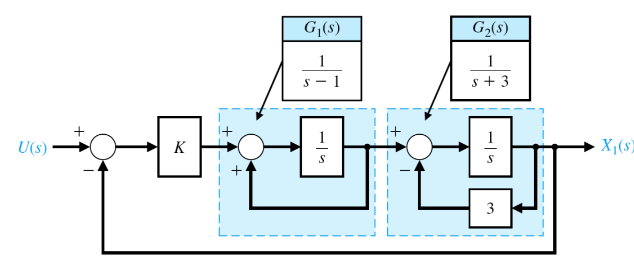


Feedback and Stability

- Feedback often improves stability.
- However, increasing the gain past a certain threshold can destabilize a system.
- **This threshold occurs when at least one root of the characteristic equation has real part equal to 0.**
- Increasing the gain can push poles from LHP to the RHP.

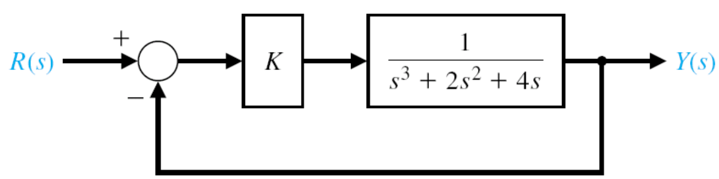
Example 2

- For what values of K will the closed-loop system be stable?



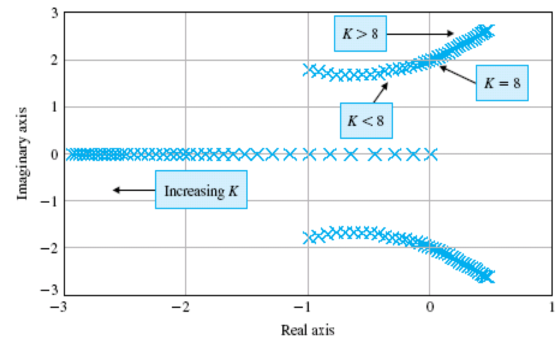
Example 3

- For what values of K will the closed-loop system be stable?



Example 3

$$0 = s^3 + 2s^2 + 4s + K$$



Example 3

$$0 = s^3 + 2s^2 + 4s + K$$

```
% This script computes the roots of the characteristic
% equation q(s) = s^3 + 2 s^2 + 4 s + K for 0<K<20
%
K=[0:0.5:20];
for i=1:length(K)
    q=[1 2 4 K(i)];
    p(:,i)=roots(q);
end
plot(real(p),imag(p),'x'), grid
xlabel('Real axis'), ylabel('Imaginary axis')
```

Loop for roots as a function of K

Summary

- Conditions for stability
  - Internal stability
  - BIBO stability
  - Marginal stability
  - Instability
- Feedback and stability
- Routh-Hurwitz stability criterion
  - Check for stability without computing roots of characteristic equation
  - Determine the range of parameters that guarantees stability