EECE 360
Lecture 18

Sketching Root Locus

Dr. Oishi

Electrical and Computer Engineering

University of British Columbia

http://courses.ece.ubc.ca/360

eece360.ubc@gmail.com

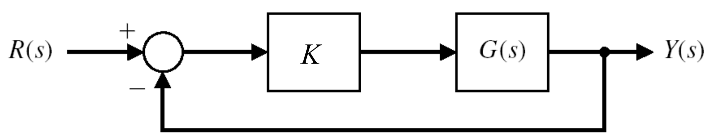
Chapter 7.2 – 7.4

Review

- Introduction to Root locus
  - The root locus plot is a graph of the locus of roots as **one** system parameter is varied
  - Evaluate zeros and poles of open-loop system to find poles of closed-loop system
- Today
  - Rules to sketch a root locus

Root Locus Method

- Consider the unit feedback system with a scalar control gain K



$$1 + KG(s) = 0$$

- The root locus originates at the poles of G(s) and terminates on the zeros of G(s).

Finding Points on the Locus

- A location s* is on the locus if
  $$1 + K\,G(s^*) = 0$$
  which is equivalent to
  $$G(s^*) = -1/K$$
- Recall that *s* is a complex number (therefore it has a *magnitude* and a *phase*) and assume K > 0

- Phase condition
  $$\text{angle}[G(s^*)]=180^{\circ}\pm 360^{\circ}n$$
  determines which points are on the locus
- The magnitude condition
  $$|G(s^*)|=1/K$$
  determines the value of K at s*

The 7 Steps to the Root Locus

- Procedure to facilitate rapid sketching of the root locus
- Locate roots of the characteristic equation in a graphical manner in the s-plane
- Roots exist on the locus for various values of the single parameter, K.

Steps to Root Locus

- ***Step 1: Prepare the root locus sketch***
  - Step 1.1: Find the char. equation 1+KG(s)=0
  - Step 1.2: Find the *m* zeros $-z_i$ and *n* poles $-p_i$ of G(s)

$$G(s^*) = \frac{k(s^*+z_1)(s^*+z_2)\ldots(s^*+z_m)}{(s^*+p_1)(s^*+p_2)\ldots(s^*+p_n)}$$

  - Step 1.3: Draw the poles and zeros of G(s) on the s-plane
  - Step 1.4: Identify number of loci (=*n*)
  - Step 1.5: Exploit symmetry across the real axis

Steps to Root Locus

- ***Step 2: Locate loci segments on the real axis.***
  - Locus lie in sections of the real axis **left of an odd number of poles and zeros**
- Step 3: Find asymptotes
  - Total of *(n-m)* asymptotes
  - **Angle** of asymptotes is

$$\phi_A = \frac{2q+1}{n-m}\cdot 180, \quad q = 0,1,\ldots,(n-m-1)$$

  - **Center** (intersection) of asymptotes is

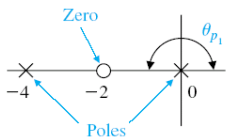
$$\sigma_A = \frac{\sum(-p_i) - \sum(-z_i)}{n-m}$$

Example: 2nd order system

- Consider the transfer function with characteristic equation

$$1 + KG(s) = 1 + \frac{2K(s+2)}{s(s+4)}$$

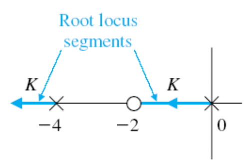
- 1. Plot the poles and zeros of KG(s)
  - Mark poles with X
  - Mark zeros with O
  - 2 poles, 1 zero -- 2 loci



(a)

Example: 2nd order system

- 2. Find root locus segments on the real line
  - Draw a line left of an odd number of **poles + zeros**
  - Check that the locus starts at poles and ends at zeros of KG(s)



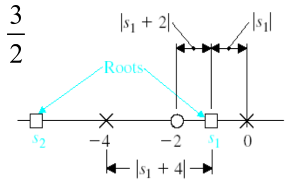
(b)

Example: 2nd order system

- Gain condition of root locus
  - To find the gain *K* for which results in a desired location s_1 is on the locus, evaluate the gain condition

$$\frac{(2K)|s_1+2|}{|s_1||s_1+4|}=1$$

$$\text{For } s_1=-1,\ K=\frac{|-1||-1+4|}{2|-1+2|}=\frac{3}{2}$$



(c)

Steps to Root Locus

- Step 2: Locate loci segments on the real axis.
  - Locus lie in sections of the real axis **left of an odd number of poles and zeros**
- ***Step 3: Find asymptotes***
  - Total of *(n-m)* asymptotes
  - **Angle** of asymptotes is

$$\phi_A=\frac{2q+1}{n-m}\cdot 180,\quad q=0,1,\ldots,(n-m-1)$$

  - **Center** (intersection) of asymptotes is

$$\sigma_A=\frac{\sum(-p_i)-\sum(-z_i)}{n-m}$$

Steps to Root Locus

- Step 3: Asymptotes:

Asymptotes of the Excess Loci: Those $n-m$ loci that don't go to z_i tend to be tangent to straight lines for $K\to\infty$.
The **angle of the asymptotes** with respect to the real axis is

$$\phi_A=\frac{(2q+1)}{n-m}180^\circ$$

q = 0, 1, 2, ..., (n-m-1)

Asymptotes Intersection Point: The linear asymptotes intersect at **one** point **on the real axis** given by

$$\sigma_A=\frac{\sum_{i=1}^{n}p_i-\sum_{j=1}^{m}z_j}{n-m}$$

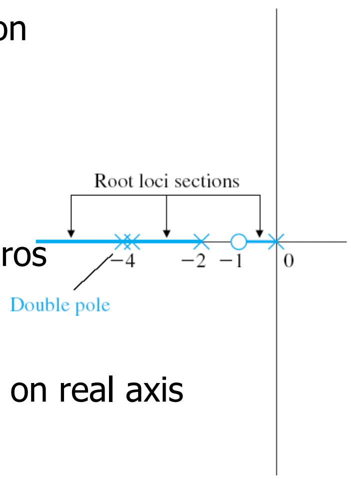
(sum of poles – sum of zeros)

Example: 4th-order system

- Characteristic equation

$$0 = 1 + KG(s)$$
$$= 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

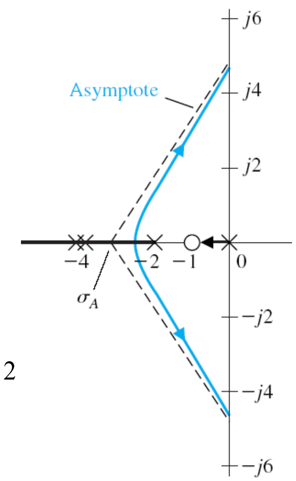
- 1. Draw poles and zeros
- 2. Find parts of locus on real axis



Example: 4th-order system

- Find asymptotes
  - $n - m = 4 - 1 = 3$
- Find center and angles

$$\sigma_A = \frac{(-2) + 2(-4) - (-1)}{4-1} = \frac{-9}{3} = -3$$
$$\phi_A = \frac{(2q+1)}{n-m}180^\circ = \frac{(2q+1)}{3}180^\circ \quad q = 0,1,2$$
$$\phi_A = 60^\circ;\ \phi_A = 180^\circ;\ \phi_A = 300^\circ;$$



Steps to Root Locus

- ***Step 4: Determine the points at which the root locus crosses the imaginary axis***
  - Evaluate using the Routh-Hurwitz criterion
- Step 5: Find breakaway points
  - Loci converge or diverge on the locus at

$$\frac{dp(s)}{ds} = 0, \quad p(s) = -\frac{1}{G(s)} \text{ (unity feedback)}$$

  - Loci approach/diverge at angles **spaced equally** about the breakaway point (and with symmetry about the real axis).

Example 3

- Consider an open-loop system

$$G(s) = \frac{1}{(s+2)^3}$$

under unity feedback.

- 1. m=0, n=3; poles at $p_1=p_2=p_3= -2$.
- 2. 1 asymptote on real axis, left of -2.
- 3. $\sigma_A = (-2)(3)/(3-0) = -2$

  $\phi_A = 60, 180, 300$
- 4. Characteristic equation

$$0 = (s+2)^3 + K = s^3 + 6s^2 + 12s + 8 + K$$

Example 3

$$0 = (s+2)^3 + K = s^3 + 6s^2 + 12s + 8 + K$$

- Routh-Hurwitz criterion:
  $a_3 = 1,\ a_2 = 6,\ a_1 = 12,\ a_0 = 8+K$
  For 0 poles in RHP,

$$0 < a_1 a_2 - a_0 a_3$$
$$< (6)(12) - (1)(8+K)$$
$$\boxed{K < 64}$$

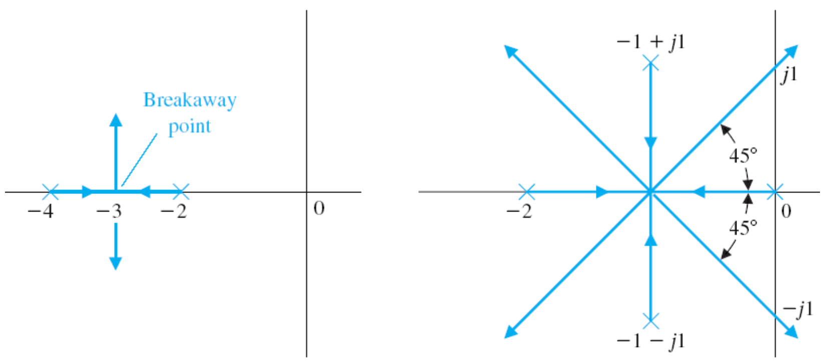
Steps to Root Locus

- Step 4: Determine the points at which the root locus crosses the imaginary axis
  - Evaluate using the Routh-Hurwitz criterion
- ***Step 5: Find breakaway points***
  - Loci converge or diverge on the real line at

$$\frac{dp(s)}{ds} = 0, \quad p(s) = -\frac{1}{G(s)} \ \text{(unity feedback)}$$

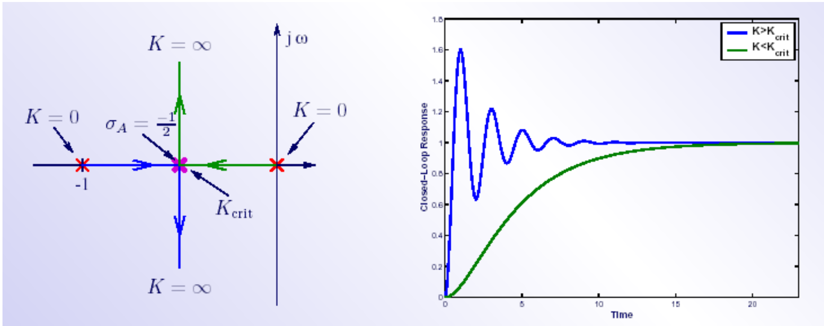
  - Loci approach/diverge at angles **spaced equally** about the breakaway point (and with symmetry about the real axis).

Breakaway Points



(a) (b)

Breakaway Points



** Each breakaway/arrival point is a point where a double (or higher order) root exists for some value of K.

Breakaway Points

- Obtaining the breakaway points

Rewriting the characteristic equation to isolate K :

$p(s) = K$

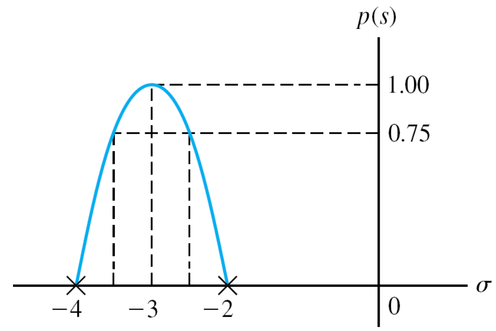
The breakaway point occur when $\dfrac{dK}{ds} = \dfrac{dp(s)}{ds} = 0$

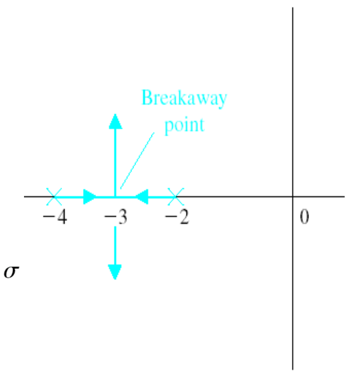
Example:

$$1 + \frac{K}{(s+2)(s+4)} = 0 \;\Rightarrow\; K = p(s) = -(s+2)(s+4)$$

$$\text{or } K = -(s^2 + 6s + 8) \;\Rightarrow\; \frac{dp(s)}{ds} = -(2s+6) = 0 \;\Rightarrow\; s = -3$$

Breakaway Points





Steps to Root Locus

- Step 6: Determine the angle of departure (from poles) and the angle of arrival (at zeros) using the **phase criterion** (e.g.,

$$180 = \sum_k \theta_{z_k} - \sum_k \theta_{p_k}$$

$$-\theta_{p_i} = 180 + \sum_{j \neq i} \theta_{p_j} - \sum_k \theta_{z_k}$$

$$\theta_{z_i} = 180 + \sum_k \theta_{p_k} - \sum_{j \neq i} \theta_{z_i}$$

- Step 7: Complete the root locus sketch.

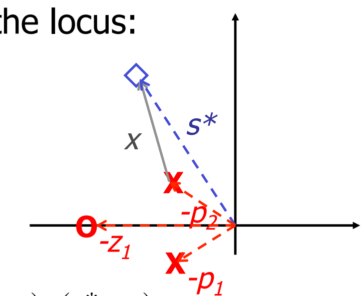
Departure angle

- Phase of a point on the locus:

$$\vec{x} + (-\vec{p}) = \vec{s}\,*$$

$$\vec{x} = \vec{p} + \vec{s}\,*$$

$$\angle(s^* + p) = \angle x$$

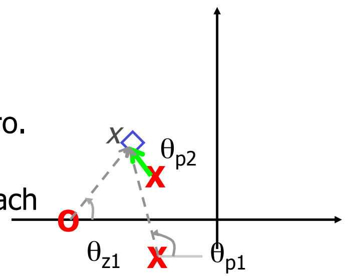


$$G(s^*) = \frac{k(s^*+z_1)(s^*+z_2)\ldots(s^*+z_m)}{(s^*+p_1)(s^*+p_2)\ldots(s^*+p_n)} =$$

$$\angle G(s^*) = \angle(s^*+z_1) + \angle(s^*+z_2) + \ldots + \angle g(s^*+z_m)$$
$$-\angle(s^*+p_1) - \angle(s^*+p_2) - \ldots - \angle(s^*+p_n)$$
$$= 180^0 \pm q \cdot 360^0$$

Departure angle

- In Step 6, evaluate the gain criterion at a point an infinitesimal distance away from a particular pole or zero.
- Compute the angles from each of the other poles (θ_{pi}) and zeros (θ_{zi}) to the particular pole/zero in question.



$$\theta_{p2} = \theta_{z1} - \theta_{p1} - 180$$

Departure angle for pole #2

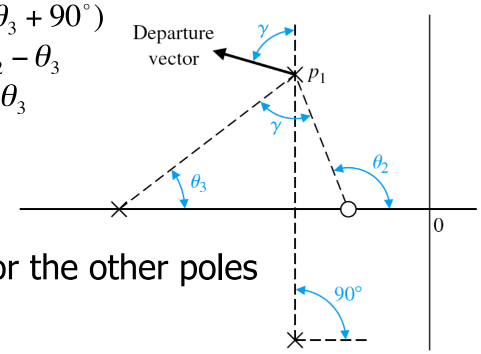
Angle from zero #1 to pole #2

Angle from zero #1 to pole #2

Example: Departure angle

- To find the departure angle θ_1 at the pole p_1, use the phase criterion

$$180^\circ = \theta_2 - (\theta_1 + \theta_3 + 90^\circ)$$
$$\theta_1 = -270^\circ + \theta_2 - \theta_3$$
$$= 90^\circ + \theta_2 - \theta_3$$



- And similarly for the other poles

Example: 3rd-order system

- Consider a system with characteristic equation

$$1+KG(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$$

$n = 3,\ m = 1$

Poles at $0, -2, -3$

Zero at -1

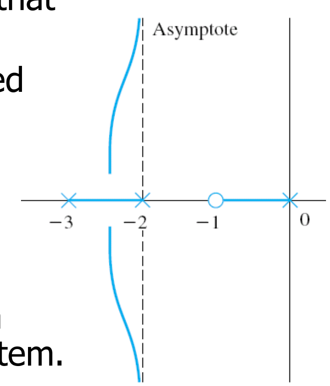
Two asymptotes at $\pm 90^\circ$

Example: 3rd-order system

$$G(s) = \frac{(s+1)}{s(s+2)(s+3)}$$

- 1. Plot poles, zeros
- 2. Find parts of locus that lie on real line
- 3. Asymptotes centered around

$$\sigma_A = \frac{(0-2-3)-(-1)}{3-1} = -\frac{4}{2} = -2$$



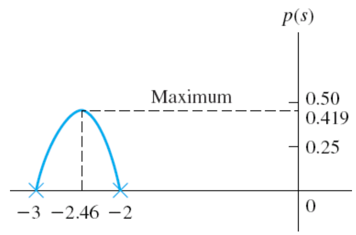
- 4. K>0 will generate a stable closed-loop system.

Example: 3rd-order system

- 5. Breakaway points

$$s(s+2)(s+3)+K(s+1)=0$$

$$p(s)=K=-\frac{s(s+2)(s+3)}{(s+1)}$$



$$\frac{dp(s)}{ds}=\frac{(s^3+5s^2+6s)-(s+1)(3s^2+10s+6)}{(s+1)^2}=0$$

$$=2s^3+8s^2+10s+6=0 \Rightarrow \boxed{s=-2.45}$$

Example: 3rd order system

- 6. Angle of departure

  Because the locus departs from the poles along the real line, the angle of departure is 0° or 180° for each pole.

  Note that when 2 poles arrive at or depart from the real line, it will always be at 90°. (Not computed in the 7 steps outlined.)

Summary

- Root locus shows evolution of closed-loop poles as **one** parameter changes
- A simple set of rules allow the loci to be sketched
- For specific plots, use Matlab tools: rlocus, rlocfind, sgrid, roots, etc.

- **Next**: Design with Root Locus