EECE 360
Lecture 19

Sketching Root Locus

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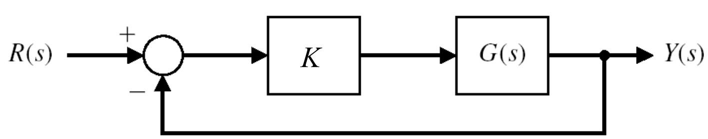
Chapter 7.2 – 7.4, 7.12

Review

- Introduction to Root locus
  - The root locus plot is a graph of the locus of roots as **one** system parameter is varied
  - Evaluate zeros and poles of open-loop system to find poles of closed-loop system
- Today
  - Rules to sketch a root locus

Root Locus Method

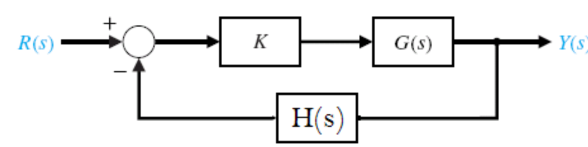
- Unity feedback



- Characteristic equation $1 + KG(s) = 0$
- The root locus originates at the poles of G(s) and terminates on the zeros of G(s).

Non-Unity Feedback

- General feedback system



- Characteristic equation

$$1 + KP(s) = 0, \quad P(s) = G(s)H(s)$$

Steps to Root Locus

- Step 1: Prepare the root locus sketch
  - Step 1.1: Find the char. equation 1+KP(s)=0
  - Step 1.2: Find the *m* zeros z_i and *n* poles p_i of P(s)

$$G(s^*) = \frac{k(s^*+z_1)(s^*+z_2)\ldots(s^*+z_m)}{(s^*+p_1)(s^*+p_2)\ldots(s^*+p_n)}$$

  - Step 1.3: Draw the poles and zeros on the s-plane
  - Step 1.4: Identify number of loci (=*n*)
  - Step 1.5: Symmetry across the real axis

Steps to Root Locus

- Step 2: Locate loci segments on the *real axis*.
  - Locus lie in sections of the real axis **left of an odd number of poles and zeros**
- Step 3: Find asymptotes
  - Total of *(n-m)* asymptotes
  - **Angle** of asymptotes is

$$\phi_A = \frac{2q+1}{n-m}\cdot 180, \quad q = 0,1,\ldots,(n-m-1)$$

  - **Center** (intersection) of asymptotes is

$$\sigma_A = \frac{\sum(-p_i)-\sum(-z_i)}{n-m}$$

Steps to Root Locus

- Step 4: Determine the points at which the root locus crosses the imaginary axis
  - Evaluate using the Routh-Hurwitz criterion
- Step 5: Find **breakaway points**
  - Loci converge or diverge on the locus at

$$\frac{dp(s)}{ds} = 0, \quad p(s) = -\frac{1}{G(s)} \text{ (unity feedback)}$$

  - Loci approach/diverge at angles **spaced equally** about the breakaway point (and with symmetry about the real axis).

Steps to Root Locus

- Step 6: Determine the angle of departure (from poles) and the angle of arrival (at zeros) using the **phase criterion** (e.g.,

$$180 = \sum_k \theta_{z_k} - \sum_k \theta_{p_k}$$

$$-\theta_{p_i} = 180 + \sum_{j\neq i} \theta_{p_j} - \sum_k \theta_{z_k}$$

$$\theta_{z_i} = 180 + \sum_k \theta_{p_k} - \sum_{j\neq i} \theta_{z_i}$$

- Step 7: Complete the root locus sketch.

Gain and Phase Criterion

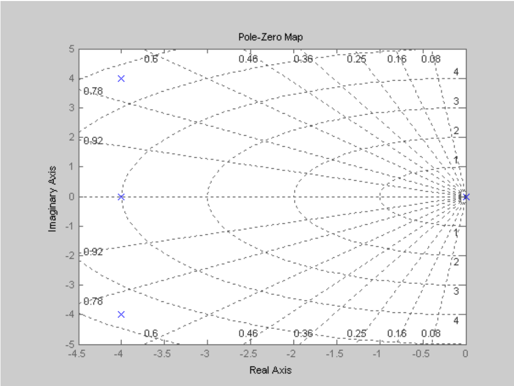
- A location s* is on the locus if
  $1 + K\,G(s^*) = 0$
  which is equivalent to
  $G(s^*) = -1/K$
- Recall that *s* is a complex number (therefore it has a *magnitude* and a *phase*) and assume K > 0

- Phase condition
  $\text{angle}[G(s^*)]=180^\circ \pm 360^\circ n$
  determines which points are on the locus
- The magnitude condition
  $|G(s^*)|=1/K$
  determines the value of K at s*

Example: Fourth-Order System

$$1+\frac{K}{s^4+12s^3+64s^2+128s}=0 \;\Rightarrow\; 1+\frac{K}{s(s+4)(s+4+4j)(s+4-4j)}=0$$

Factor G(s)



Want to plot the root-locus as K varies for K>0.

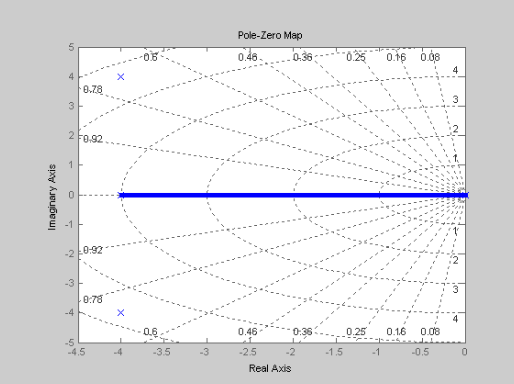
Fourth-Order System

Step 1:

Locate the poles

$n = 4$ and $m = 0$ implies that there are 4 infinite zeros.

$n = 4$ implies that there are 4 separate loci



Step 2:

Locate the seg. of the real axis.

Fourth-Order System

Step 3: locate the asymptotes

Asymptotes:

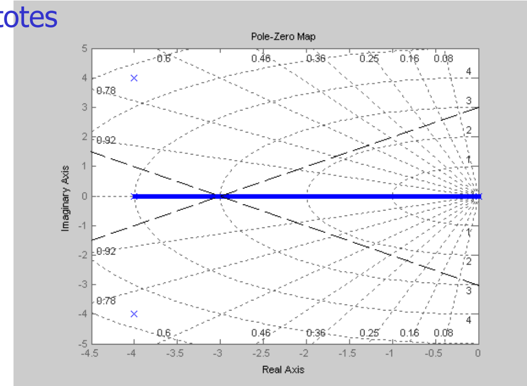
Angles:

$$\phi_A = \frac{(2q+1)}{4}180^\circ, \quad q = 0,1,2,3$$

$$\phi_A = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

Centroid:

$$\sigma_A = \frac{-4-4-4j-4+4j}{4} = -3$$



Fourth-Order System

Step 4: Determine the points crossing the imaginary axis

- Intersection with imaginary axis

$$s^4 + 12s^3 + 64s^2 + 128s + K = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array}$$

$$\Rightarrow b_1 s^2 + K = 0 \Rightarrow 53.33(s + 3.266j)(s - 3.266j) = 0$$

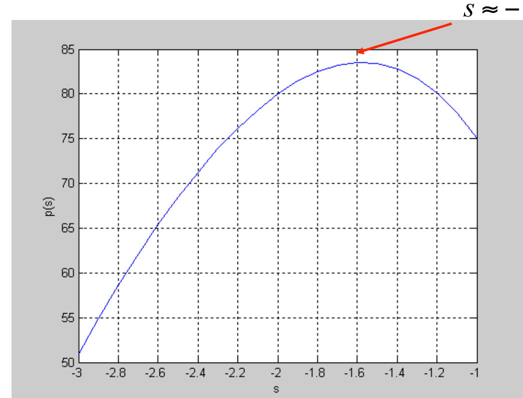
$$b_1 = \frac{12(64) - 128}{12} = 53.33 \text{ and } c_1 = \frac{53.33(128) - 12K}{53.33} \Rightarrow K = 568.89$$

Fourth-Order System

Step 5: Determine the breakaway point

Breakaway point: $K = p(s) = -s^4 - 12s^3 - 64s^2 - 128s$

$s \approx -1.6$



Fourth-Order System

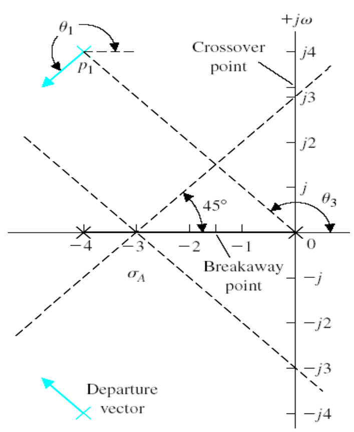
Step 6: Determine the angles of departure/arrival

- Angle of departure:

Angle of departure at pole p_1 :

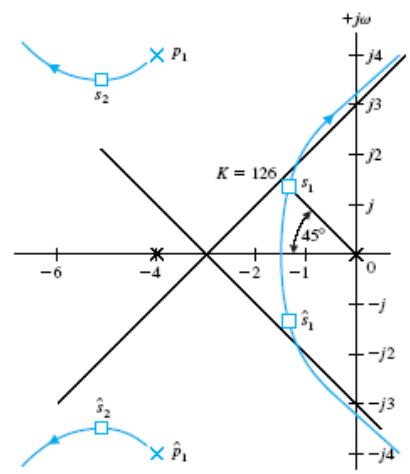
$\theta_1 + 90^\circ + 90^\circ + \theta_3 = 180^\circ$

Because $\theta_3 = 135^\circ$, $\theta_1 = -135^\circ = 225^\circ$

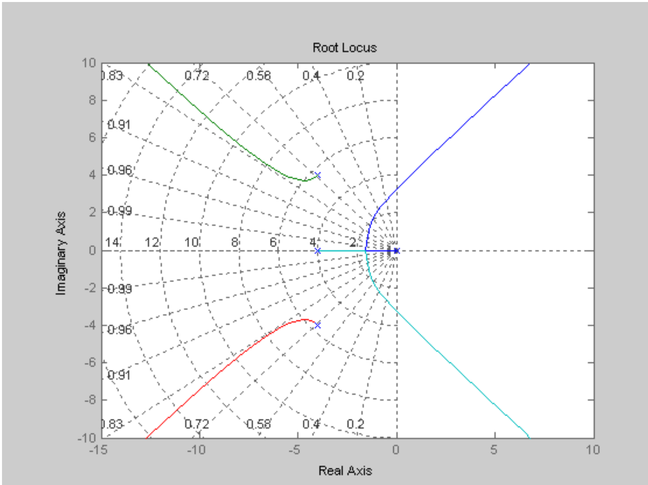


Fourth-Order System

- Angle of arrival:

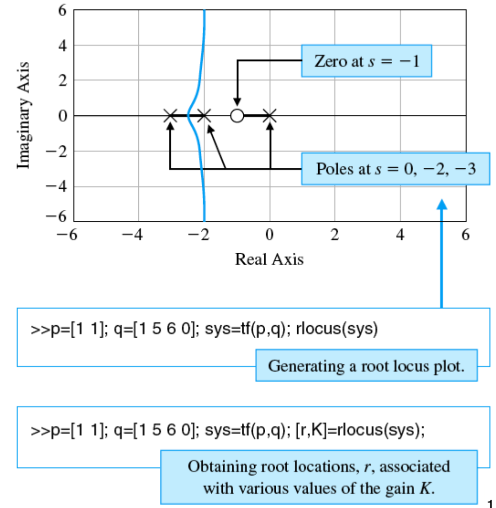


Fourth-Order System



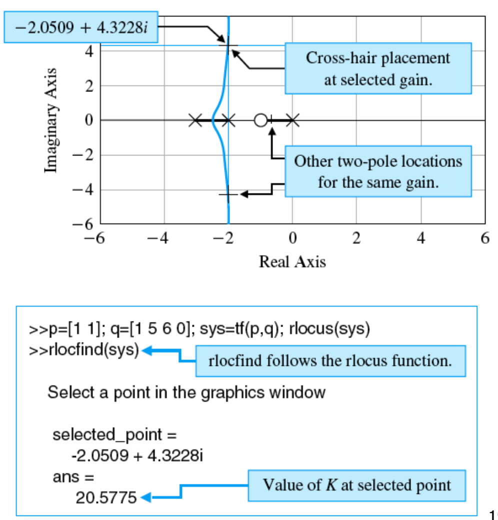
Using Matlab

- 'rlocus'



Using Matlab

- 'rlocus'
- 'rlocfind'

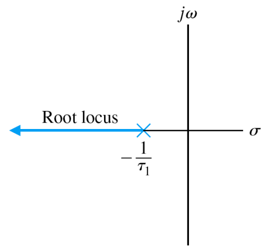


Typical Root Locus Plots

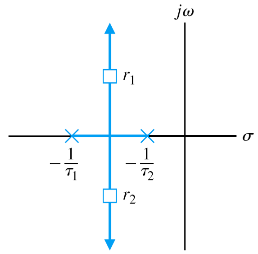
- Table 7.2, Dorf and Bishop, for general sketching guidelines
- Table 7.7, Dorf and Bishop, for generic diagrams of some common systems

Typical Root Locus Plots

- First-order systems

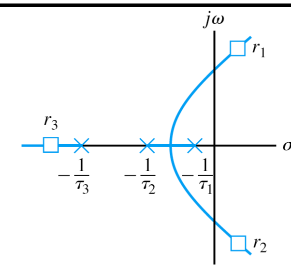


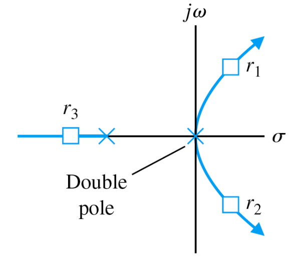
- Second-order systems



Typical Root Locus Plots

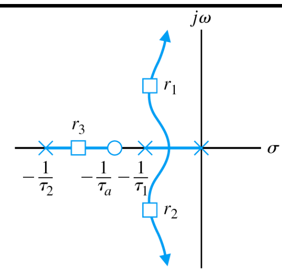
- Third-order systems



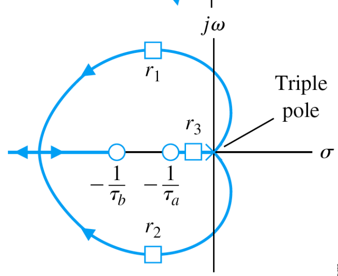


Typical Root Locus Plots

- Third-order system with one zeros



- Third-order system with two zeros



Root Locus for Design

- Location of roots indicates performance of the closed-loop system.
- Move roots along the root locus by varying K.
- Pick K to accommodate transient or steady-state constraints (e.g. desired damping ratio, natural frequency, overshoot, etc.)
- Poles closest to the imaginary axis are **dominant** because they are slowest (e.g. they take the longest to die out).

Summary

- Complete root locus sketches
  - Number of poles, zeros, and asymptotes
  - Segments on the real axis
  - Center/angle of asymptotes
  - Departure/arrival angles
  - Imaginary axis crossings

- Design with root locus
  - Choosing K to meet transient/steady-state response criteria

- **Next**: PID with Root Locus