EECE 360
Lecture 21

Frequency Response: Bode Diagrams

Dr. Oishi

Electrical and Computer Engineering

University of British Columbia

http://courses.ece.ubc.ca/360

eece360.ubc@gmail.com

Chapter 8.1-8.3

Context

- Modeling LTI systems
  - Transfer functions
  - State-space
- Analysis and control of LTI systems
  - Root locus
  - **Bode (this week)**
  - Nyquist
  - State-space methods

Frequency Response Plots

- See Dorf, Appendix G (online) for review of complex numbers
- Transfer function G(s) in frequency domain:

$$G(j\omega) = G(s)\big|_{s=j\omega}$$
$$= |G(j\omega)|\, e^{j\phi(\omega)}$$
$$= |G(j\omega)|\, \angle\phi(\omega)$$
$$= R(\omega) + jX(\omega)$$

$$|G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2$$

$$\angle\phi(j\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)}$$

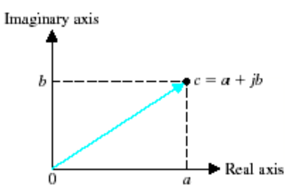


FIGURE G.1 Rectangular form of a complex number.

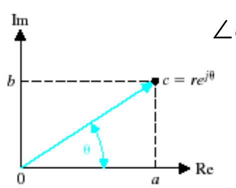


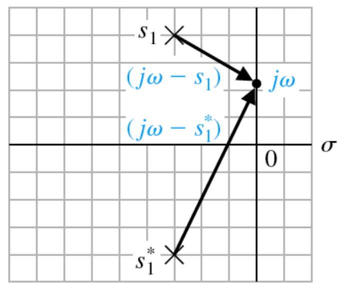
FIGURE G.2 Exponential form of a complex number.

Frequency Response Methods

- **Frequency response:** The steady-state response of the system to a sinusoidal input as the frequency varies.
- Examine the transfer function G(s) when s = jω.
- Consider a test input, *r(t) = A sin(ωt)*
- The output is $y(t) = A|G(j\omega)|\sin(\omega t + \phi)$, $\phi = \angle G(j\omega)$ which has
  - **Same frequency as input**
  - **Different magnitude than input**
  - **Different phase than input**

Frequency Response

- **Frequency response:** The steady-state response of the system to a sinusoidal input as the frequency varies.
- Another interpretation: The frequency response is the sum of log magnitude and phases of vectors from poles and zeros of G(s) to the point jω on the imaginary axis



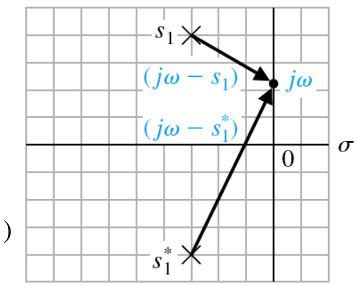
Frequency Response

- **Frequency response:** The steady-state response of the system to a sinusoidal input as the frequency varies.
- Consider the system

$$G(s) = \frac{\omega_n^2}{(j\omega - s_1)(j\omega - s_1^*)}$$

$$|G(s)| = \frac{\omega_n^2}{|j\omega - s_1||j\omega - s_1^*|}$$

$$\angle G(s) = -\angle(j\omega - s_1) - \angle(j\omega - s_1^*)$$



Hendrik Wade Bode

- 1905-1982, USA
- PhD from Columbia in 1935
- Entire career at Bell Labs
- Invented magnitude and phase frequency plots in 1938
- Many other contributions in electrical engineering and control

Laplace vs. Fourier Transform

Laplace transform:

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \quad \text{where } s = \sigma + j\omega$$

Fourier transform:

$$F(j\omega) = L\{f(t)\} = \int_0^\infty f(t)e^{-j\omega t}dt$$

Setting $s = j\omega$ in $F(s)$ yields the Fourier transform of $f(t)$

(But Fourier Transform is often used for signals that exist for t<0)

Bode Diagram

- Logarithmic plot of gain and phase of a transfer function G(s) in the frequency domain G(jω)
- Two plots
  - **Logarithmic gain** *20 $\log_{10}|G(j\omega)|$* (dB) vs. $\log_{10}\omega$
  - **Phase angle** *$\phi(j\omega)$* (degrees) vs. $\log_{10}\omega$
- Recall that since $\log(ab) = \log a + \log b$

$$\angle(ab) = \angle a + \angle b$$

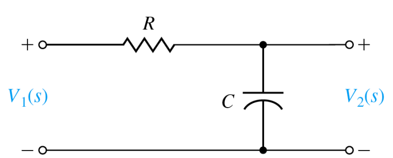
effects of poles and zeros is **additive** in Bode diagrams.

Bode Diagram

- On a Bode plot, for simple poles and zeros, slopes are **approximately linear**
  - Log gain - Log frequency
  - Phase - Log frequency
- General procedure:
  - Start a low frequencies
  - Identify break points
  - Approximate slopes before and after break points
  - Effect is cumulative as frequency increases, for gain and phase

Bode Diagram: RC Filter

- Example: RC filter



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{sRC+1}$$

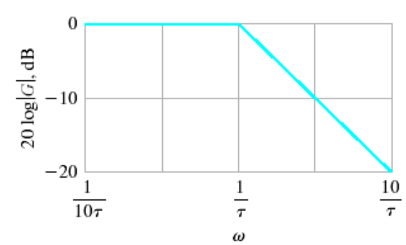
$$G(j\omega) = \frac{1}{j\omega\tau + 1} \qquad (\tau = RC)$$

$$20\log|G| = 20\log\left(\frac{1}{1+(\omega\tau)^2}\right)^{\frac{1}{2}} = -10\log\left(1+(\omega\tau)^2\right)$$

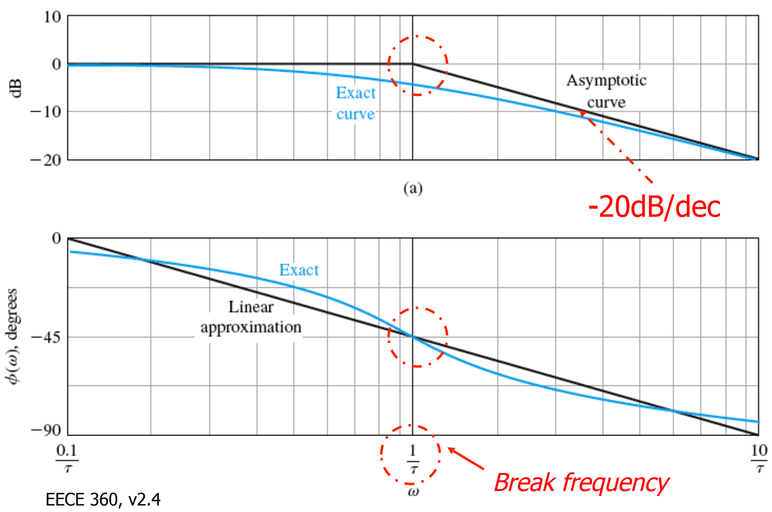
$$\phi(\omega) = -\tan^{-1}\omega\tau$$

Bode Diagram: RC Filter

- Gain
  - For ω « 1/t, |G(jω)| ≈ -10 log(1) = 0 dB
  - For ω » 1/t, |G(jω)| ≈ -10 log($(\omega\tau)^2$) = -20 log(ωτ) dB
  - At ω = 1/t, |G(jω)| = -10 log(2) ≈ -3.01 dB
- Phase
  - angle(G(jω)) ≈ $-\tan^{-1}(\omega\tau)$
  - At ω = 1/τ,
    angle(G(jω)) = $-\tan^{-1}(1)$
    = -45^0



Bode Diagram: RC Filter



Bode Diagram

- Standard components in a transfer function
  - Constant gain *K*
  - Poles (or zeros) at the origin $(j\omega)$
  - Poles (or zeros) on the real axis $(j\omega\tau+1)$
  - Complex conjugate poles (or zeros) $(1+(2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2\ )$
- Use generic Bode plots of each of these components as "building blocks" to determine Bode plots of more complicated transfer functions

Bode Diagram

- Decompose a high order transfer function into a product of simple standard components
- These components are **additive** in Bode plots
- This is the main advantage of using logarithmic plots

Bode Diagram

- **1. Constant gain *K***
- Transfer function

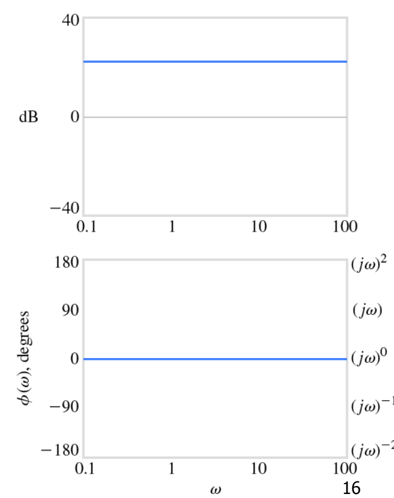
$$G(s) = K$$

- Log gain

$$20\log|G(j\omega)| = 20\log K$$

- Phase

$$\angle G(j\omega) = 0^0$$



Bode Diagram

- **2. Poles (or zeros) at the origin (jω)**
- Transfer function

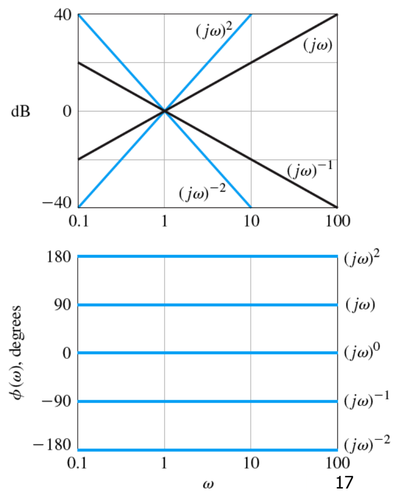
$$G(s) = \frac{1}{s^N}$$

- Log gain

$$20\log|G(j\omega)| = -20N\log\omega$$

- Phase

$$\angle G(j\omega) = -90^0 N$$



Bode Diagram

- **3. Poles (or zeros) on the real axis (jωτ+1)**
- Transfer function

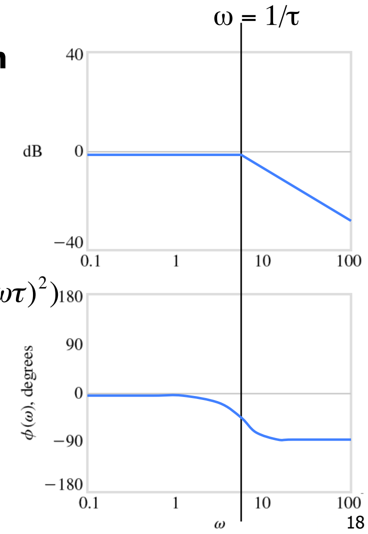
$$G(s) = \frac{1}{s\tau + 1}$$

- Log gain

$$20\log|G(j\omega)| = -10\log(1 + (\omega\tau)^2)$$

- Phase

$$\angle G(j\omega) = -\tan^{-1}(\omega\tau)$$



Bode Diagram

- **4. Complex conjugate poles (or zeros) $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)$**
- Transfer function

$$G(s) = \frac{1}{1 + 2\zeta/\omega_n\, s + (s/\omega_n)^2}$$

- Log gain

$$20\log|G(j\omega)| = -10\log((1 - \tfrac{\omega}{\omega_n})^2 + 4(\zeta\tfrac{\omega}{\omega_n})^2)$$

- For $\omega/\omega_n \ll 1$, $|G(j\omega)| \approx 0$
- For $\omega/\omega_n \gg 1$, $|G(j\omega)| \approx -10 \log (\omega/\omega_n)^4$ $= -40 \log (\omega/\omega_n)$

Bode Diagram

- **4. Complex conjugate poles (or zeros) $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)$**
- Transfer function

$$G(s) = \frac{1}{1 + 2\zeta/\omega_n\, s + (s/\omega_n)^2}$$

- Log gain

$$20\log|G(j\omega)| = -10\log((1 - \tfrac{\omega}{\omega_n})^2 + 4(\zeta\tfrac{\omega}{\omega_n})^2)$$

- Phase

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2}\right)$$

Bode Diagram

- **4. Complex conjugate poles (or zeros) $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)$**
- Actually, the maximum value $M_{p\omega}$ of the frequency response

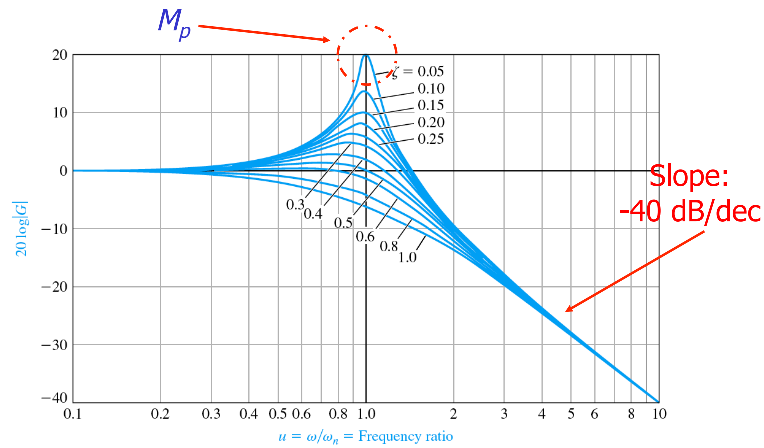
$$M_{p\omega} = |G(\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

occurs at the resonant frequency $\omega = \omega_r$.

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Bode Diagram

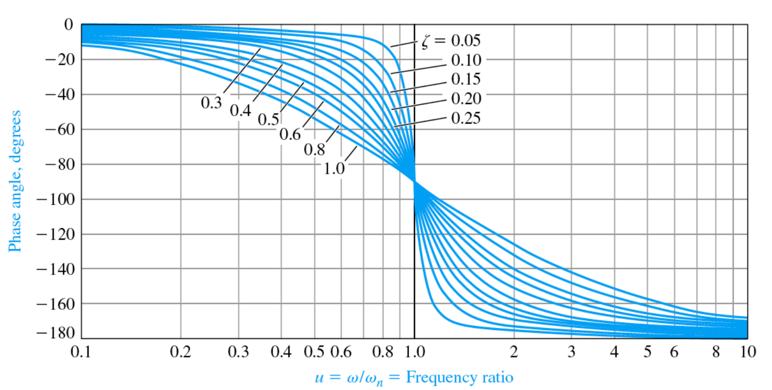
$$G(s) = \frac{1}{1 + 2\zeta/\omega_n\, s + (s/\omega_n)^2}$$



Note that for sketching $\omega_r \approx \omega_d$, the damped frequency

Bode Diagram

$$G(s) = \frac{1}{1 + 2\zeta/\omega_n\, s + (s/\omega_n)^2}$$



Sketching Bode Diagram

Generalized $G(j\omega) = \dfrac{K\prod_i (1+j\omega\tau_i)}{(j\omega)^N \prod_m (1+j\omega\tau_m)\prod_k [1+(2\xi_k/\omega_{n_k})j\omega + (j\omega/\omega_{n_k})^2]}$,

So, we have

$$20\log(G(\omega)) = 20\log(K_b) + 20\sum_i \log|1+j\omega\tau_i|$$

$$-20\mathrm{N}\log(j\omega) - 20\sum_m \log|1+j\omega\tau_i| - 20\sum_k \log|1+(2\xi_k/\omega_{n_k})j\omega + (j\omega/\omega_{n_k})^2|;$$

$$\phi(\omega) = \sum_i \tan^{-1}(\omega\tau_i) - N(90^\circ) - \sum_k \tan^{-1}(\omega\tau_k) - \sum_k \tan^{-1}\left(\frac{2\xi_k\omega_{n_k}\omega}{\omega_{n_k}^2 - \omega^2}\right)$$

Example: Sketching Bode

4 different components:

1. Constant gain K;
2. Poles (zeros) at origin (j ω);
3. Poles (zeros) on the real axis (jω τ +1);
4. Complex conjugate poles (zeros)

$$G(s) = \frac{5(1+0.1s)}{s(1+0.5s)(1+0.6s/50+s^2/2500)}$$

1 4 2 3 5

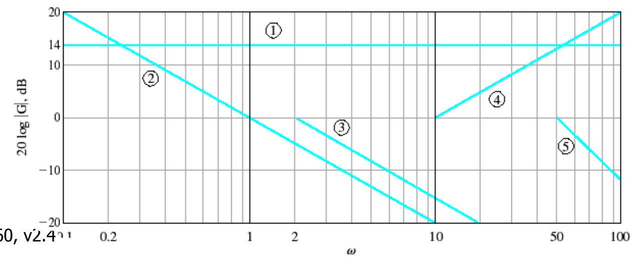
Gain $K = 5$ (14dB)

Pole at origin

Pole at $\omega = 2$

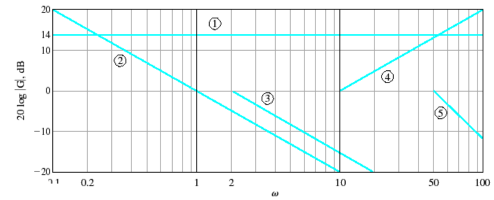
Zero at ω=10

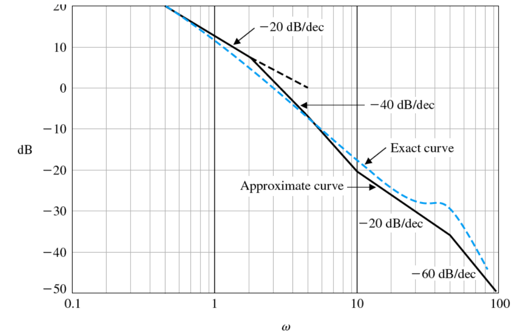
Complex poles at ω=50



Example: Sketching Bode

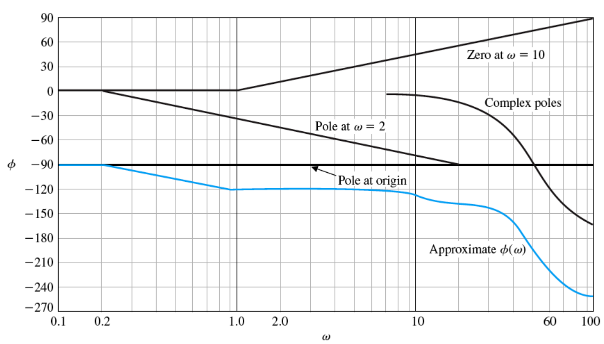
- Gain





Example: Sketching Bode

- Phase



Non-Minimum Phase System

- A transfer function is called **minimum phase** if all its zeros lie in the left-hand plane.

$$G(s) = \frac{s+z}{s+p}, \quad p,z > 0$$

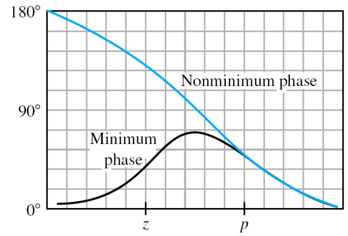
- It is called **non-minimum phase** if it has any zeros in the right-hand plane.

$$G(s) = \frac{s-z}{s+p}, \quad p,z > 0$$

- Recall that the frequency response only exists for stable systems (poles in LHP)

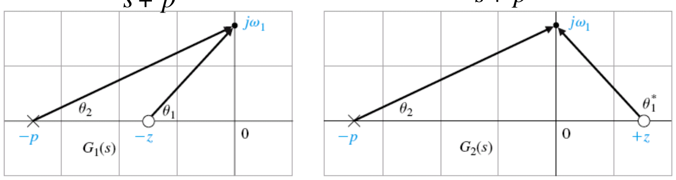
Non-Minimum Phase System

- Same magnitude,
  different phase



$$G(s) = \frac{s+z}{s+p}, \quad p,z>0$$

$$G(s) = \frac{s-z}{s+p}, \quad p,z>0$$



(a) (b)

Summary

- Response of linear system to sinusoidal input is sine wave of different magnitude and phase
- Bode diagram
- Bode plots of common components
- Sketching the Bode diagram
- Minimum phase vs Non-minimum phase