

# Nyquist Stability Criterion

A stability test for time invariant linear systems can also be derived in the frequency domain. It is known as Nyquist stability criterion. It is based on the complex analysis result known as *Cauchy's principle of argument*. Note that the system transfer function is a complex function. By applying Cauchy's principle of argument to the *open-loop system* transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the Nyquist stability criterion (Nyquist, 1932).

The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques.

We present only the essence of the Nyquist stability criterion and define the phase and gain stability margins. The Nyquist method is used for studying the stability of linear systems with pure time delay.

For a SISO feedback system the closed-loop transfer function is given by

$$M(s) = \frac{G(s)}{1 + H(s)G(s)}$$

where  $G(s)$  represents the system and  $H(s)$  is the feedback element. Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation

$$1 + H(s)G(s) = 0 = \Delta(s)$$

which, in fact, represents the *system characteristic equation*.

In the following we consider the complex function

$$D(s) = 1 + H(s)G(s)$$

whose zeros are the closed-loop poles of the transfer function. In addition, it is easy to see that the poles of  $D(s)$  are the zeros of  $M(s)$ . At the same time the poles of  $D(s)$  are the open-loop control system poles since they are contributed by the poles of  $H(s)G(s)$ , which can be considered as the open-loop control system transfer function—obtained when the feedback loop is open at some point. The Nyquist stability test is obtained by applying the Cauchy principle of argument to the complex function  $D(s)$ . First, we state Cauchy's principle of argument.

## Cauchy's Principle of Argument

Let  $F(s)$  be an analytic function in a closed region of the complex plane  $s$  given in Figure 4.6 except at a finite number of points (namely, the poles of  $F(s)$ ). It is also assumed that  $F(s)$  is analytic at every point on the contour. Then, *as  $s$  travels around the contour in the  $s$ -plane in the clockwise direction, the function  $F(s)$  encircles the origin in the  $(\operatorname{Re}\{F(s)\}, \operatorname{Im}\{F(s)\})$ -plane in the same direction  $N$  times (see Figure 4.6), with  $N$  given by*

$$N = Z - P$$

*where  $Z$  and  $P$  stand for the number of zeros and poles (including their multiplicities) of the function  $F(s)$  inside the contour.*

The above result can be also written as

$$\arg \{F(s)\} = (Z - P)2\pi = 2\pi N$$

which justifies the terminology used, “the principle of argument”.

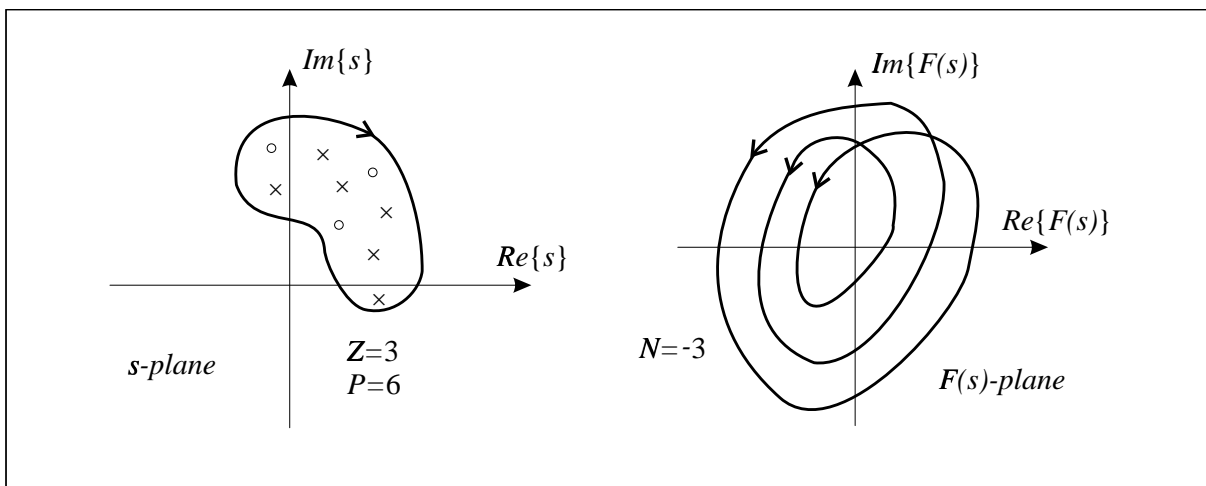


Figure 4.6: Cauchy's principle of argument

## Nyquist Plot

The Nyquist plot is a polar plot of the function  $D(s) = 1 + G(s)H(s)$  when  $s$  travels around the contour given in Figure 4.7.

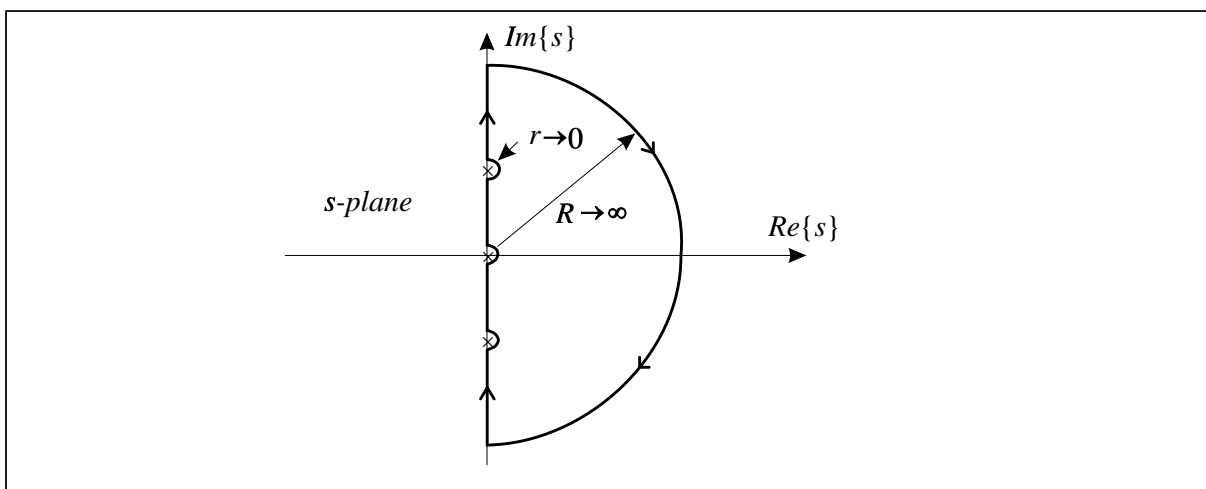


Figure 4.7: Contour in the  $s$ -plane

The contour in this figure covers the whole unstable half plane of the complex plane  $s$ ,  $R \rightarrow \infty$ . Since the function  $D(s)$ , according to Cauchy's principle of argument, must be analytic at every point on the

contour, the poles of  $D(s)$  on the imaginary axis must be encircled by infinitesimally small semicircles.

### **Nyquist Stability Criterion**

It states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot of the complex function  $D(s)$ .

This can be easily justified by applying Cauchy's principle of argument to the function  $D(s)$  with the  $s$ -plane contour given in Figure 4.7. Note that  $Z$  and  $P$  represent the numbers of zeros and poles, respectively, of  $D(s)$  in the unstable part of the complex plane. At the same time, *the zeros of  $D(s)$  are the closed-loop system poles, and the poles of  $D(s)$  are the open-loop system poles (closed-loop zeros).*

The above criterion can be slightly simplified if instead of plotting the function  $D(s) = 1 + G(s)H(s)$ , we plot only the function  $G(s)H(s)$  and count encirclement of the Nyquist plot of  $G(s)H(s)$  around the point  $(-1, j0)$ , so that the modified Nyquist criterion has the following form.

*The number of unstable closed-loop poles ( $Z$ ) is equal to the number of unstable open-loop poles ( $P$ ) plus the number of encirclements ( $N$ ) of the point  $(-1, j0)$  of the Nyquist plot of  $G(s)H(s)$ , that is*

$$Z = P + N$$



## Phase and Gain Stability Margins

Two important notions can be derived from the Nyquist diagram: *phase and gain stability margins*. The phase and gain stability margins are presented in Figure 4.8.

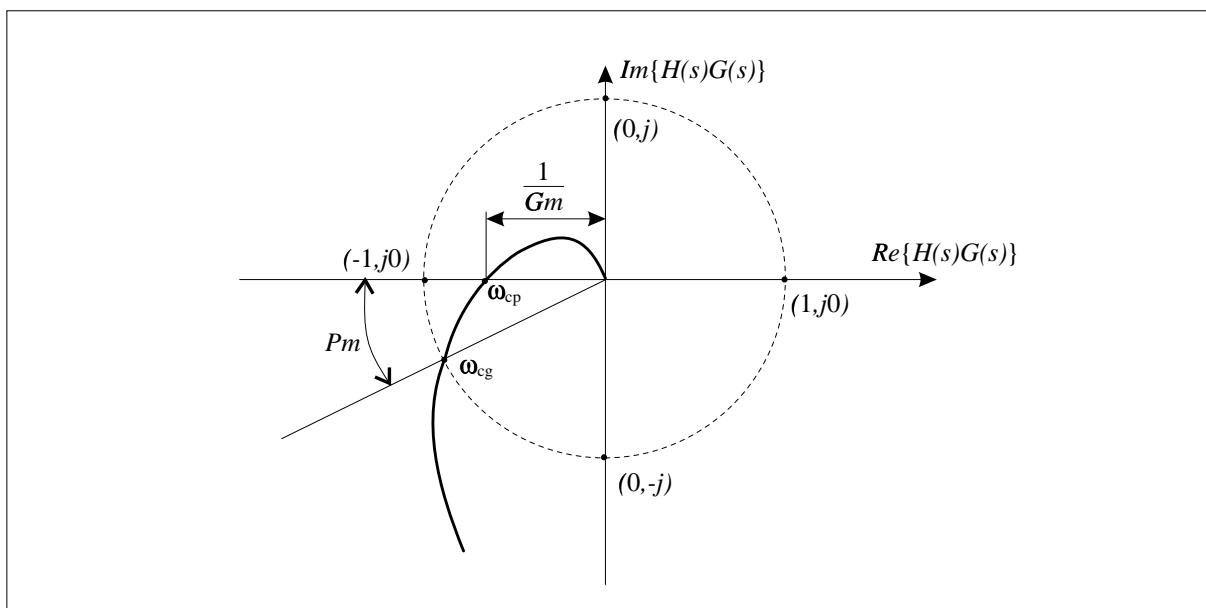


Figure 4.8: Phase and gain stability margins

They give the degree of relative stability; in other words, they tell how far the given system is from the instability region. Their formal definitions are given by

$$Pm = 180^\circ + \arg \{G(j\omega_{cg})H(j\omega_{cg})\}$$

$$Gm [dB] = 20 \log \frac{1}{|G(j\omega_{cp})H(j\omega_{cp})|} [dB]$$

where  $\omega_{cg}$  and  $\omega_{cp}$  stand for, respectively, the *gain and phase crossover frequencies*, which from Figure 4.8 are obtained as

$$|G(j\omega_{cg})H(j\omega_{cg})| = 1 \Rightarrow \omega_{cg}$$

and

$$\arg \{G(j\omega_{cp})H(j\omega_{cp})\} = 180^\circ \Rightarrow \omega_{cp}$$

**Example 4.23:** Consider a control system represented by

$$G(s)H(s) = \frac{1}{s(s+1)}$$

Since this system has a pole at the origin, the contour in the  $s$ -plane should encircle it with a semicircle of an infinitesimally small radius. This contour has three parts (a), (b), and (c). Mappings for each of them are considered below.

(a) On this semicircle the complex variable  $s$  is represented in the polar form by  $s = Re^{j\Psi}$  with  $R \rightarrow \infty$ ,  $-\frac{\pi}{2} \leq \Psi \leq \frac{\pi}{2}$ . Substituting  $s = Re^{j\Psi}$  into  $G(s)H(s)$ , we easily see that  $G(s)H(s) \rightarrow 0$ . Thus, the huge semicircle from the  $s$ -plane maps into the origin in the  $G(s)H(s)$ -plane (see Figure 4.9).

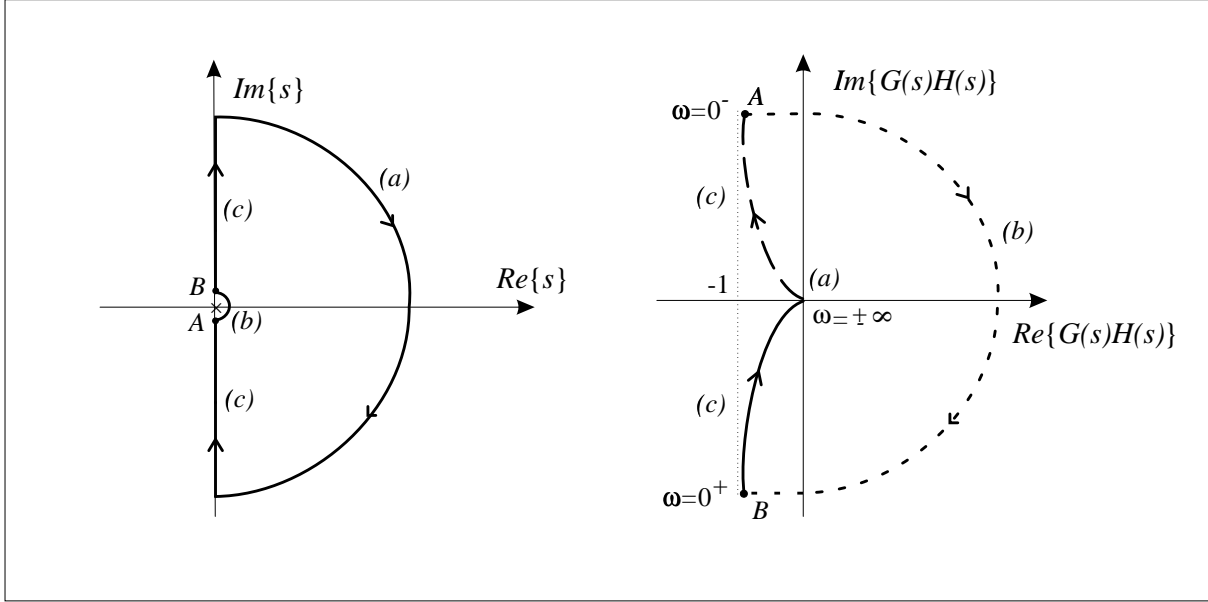


Figure 4.9: Nyquist plot for Example 4.23

(b) On this semicircle the complex variable  $s$  is represented in the polar form by  $s = re^{j\Phi}$  with  $r \rightarrow 0$ ,  $-\frac{\pi}{2} \leq \Phi \leq \frac{\pi}{2}$ , so that we have

$$G(s)H(s) \rightarrow \frac{1}{re^{j\Phi}} \rightarrow \infty \times \arg(-\Phi)$$

Since  $\Phi$  changes from  $-\frac{\pi}{2}$  at point A to  $\frac{\pi}{2}$  at point B,  $\arg\{G(s)H(s)\}$

will change from  $\frac{\pi}{2}$  to  $-\frac{\pi}{2}$ . We conclude that the infinitesimally small semicircle at the origin in the  $s$ -plane is mapped into a semicircle of infinite radius in the  $G(s)H(s)$ -plane.

(c) On this part of the contour  $s$  takes pure imaginary values, i.e.  $s = j\omega$  with  $\omega$  changing from  $-\infty$  to  $+\infty$ . Due to symmetry, it is sufficient to study only mapping along  $0^+ \leq \omega \leq +\infty$ . We can find the real and imaginary parts of the function  $G(j\omega)H(j\omega)$ , which are given by

$$\begin{aligned} \operatorname{Re}\{G(j\omega)H(j\omega)\} &= \frac{-1}{\omega^2 + 1} \\ \operatorname{Im}\{G(j\omega)H(j\omega)\} &= \frac{-1}{\omega(\omega^2 + 1)} \end{aligned}$$

From these expressions we see that neither the real nor the imaginary parts can be made zero, and hence the Nyquist plot has no points of intersection with the coordinate axis. For  $\omega = 0^+$  we are at point B and since the plot at  $\omega = +\infty$  will end up at the origin, the Nyquist diagram corresponding to part (c) has the form as shown in Figure 4.9. Note that the vertical asymptote of the Nyquist plot in Figure 4.9 is given by  $\text{Re}\{G(j0^\pm)H(j0^\pm)\} = -1$  since at those points  $\text{Im}\{G(j0^\pm)H(j0^\pm)\} = \mp\infty$ .

From the Nyquist diagram we see that  $N = 0$  and since there are no open-loop poles in the left half of the complex plane, i.e.  $P = 0$ , we have  $Z = 0$  so that the corresponding closed-loop system has no unstable poles.

The Nyquist plot is drawn by using the MATLAB function `nyquist`

```
num=1; den=[1 1 0];  
nyquist(num,den);  
axis([-1.5 0.5 -10 10]);  
axis([-1.2 0.2 1 1]);
```

The MATLAB Nyquist plot is presented in Figure 4.10. It can be seen from Figures 4.8 and 4.9 that  $1/Gm = 0$ , which implies that  $Gm = \infty$ . Also, from the same figures it follows that  $\omega_{cp} = \infty$ . In order to find the phase margin and the corresponding gain crossover frequency we use the MATLAB function `margin` as follows

```
[Gm,Pm,wcp,wcg]=margin(num,den)
```

producing, respectively, gain margin, phase margin, phase crossover frequency, and gain crossover frequency. The required phase margin and gain crossover frequency are obtained as  $Pm = 53.4108^\circ$ ,  $\omega_{cg} = 0.7862$  rad/s.

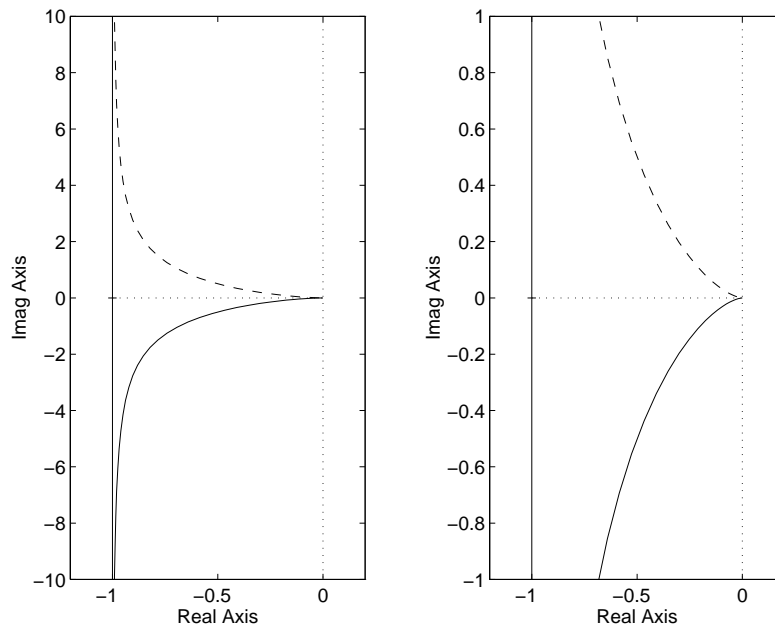


Figure 4.10: MATLAB Nyquist plot for Example 4.23



**Example 4.24:** Consider now the following system, obtained from the one in the previous example by adding a pole, that is

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

The contour in the  $s$ -plane is the same as in the previous example. For cases (a) and (b) we have the same analyses and conclusions. It remains to examine case (c). If we find the real and imaginary parts of  $G(j\omega)H(j\omega)$ , we get

$$\begin{aligned} \operatorname{Re}\{G(j\omega)H(j\omega)\} &= \frac{-3}{9\omega^2 + (2 - \omega^2)^2} \\ \operatorname{Im}\{G(j\omega)H(j\omega)\} &= \frac{-(2 - \omega^2)}{\omega[9\omega^2 + (2 - \omega^2)^2]} \end{aligned}$$

It can be seen that an intersection with the real axis happens at  $\omega = \sqrt{2}$  at the point  $\text{Re}\{G(j\sqrt{2})H(j\sqrt{2})\} = -1/6$ . The Nyquist plot is given in Figure 4.11. The corresponding Nyquist plot obtained by using MATLAB is given in Figure 4.12.

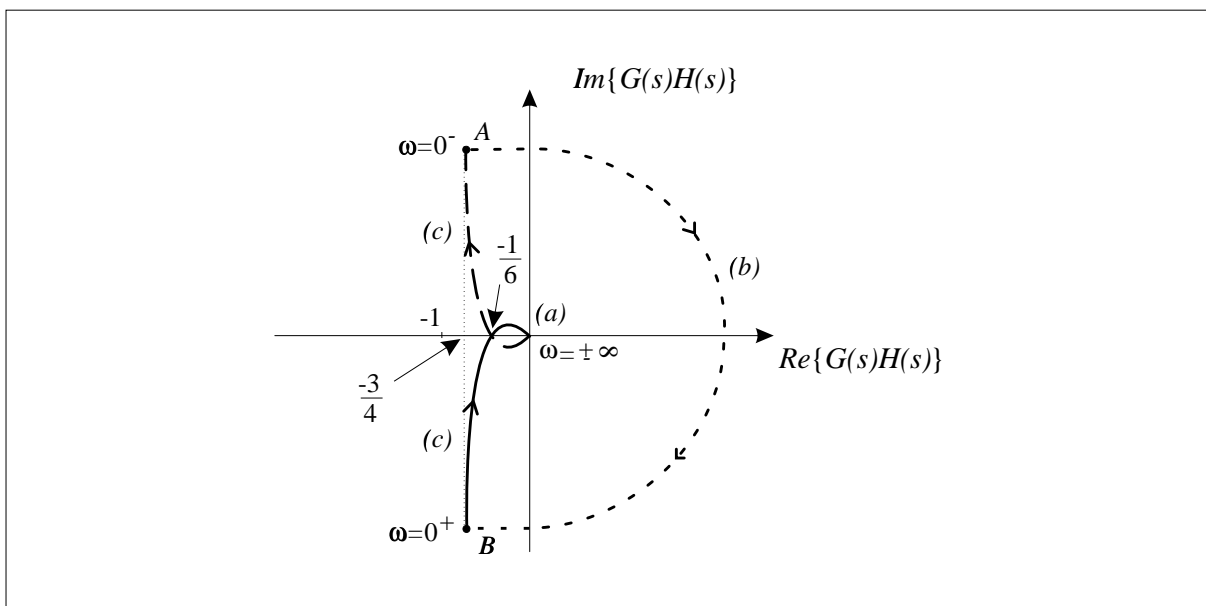


Figure 4.11: Nyquist plot for Example 4.24

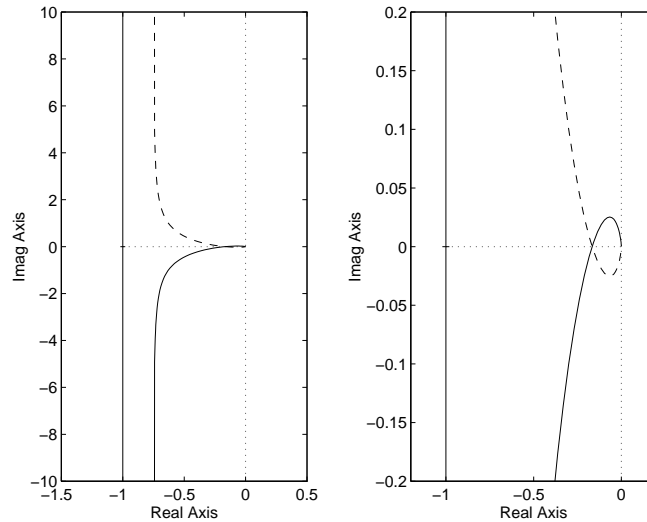


Figure 4.12: MATLAB Nyquist plot for Example 4.24

Note that the vertical asymptote is given by  $\text{Re}\{G(j0)H(j0)\} = -3/4$ . Thus, we have  $N = 0$ ,  $P = 0$ , and  $Z = 0$  so that the closed-loop system is stable. The MATLAB function `margin` produces

$$Gm = 6 \text{ dB}, \quad Pm = 53.4108^\circ$$

$$\omega_{cg} = 0.4457 \text{ rad/s}, \quad \omega_{cp} = 1.4142 \text{ rad/s}$$

