VARIABLE-FREQUENCY NETWORK PERFORMANCE

LEARNING GOALS

Variable-Frequency Response Analysis
- Network performance as function of frequency.
- Transfer function

Sinusoidal Frequency Analysis
- Bode plots to display frequency response data

Resonant Circuits
- The resonance phenomenon and its characterization

Scaling
- Impedance and frequency scaling

Filter Networks
- Networks with frequency selective characteristics: low-pass, high-pass, band-pass

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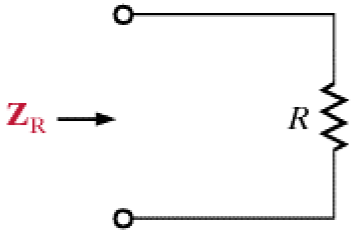
VARIABLE FREQUENCY-RESPONSE ANALYSIS

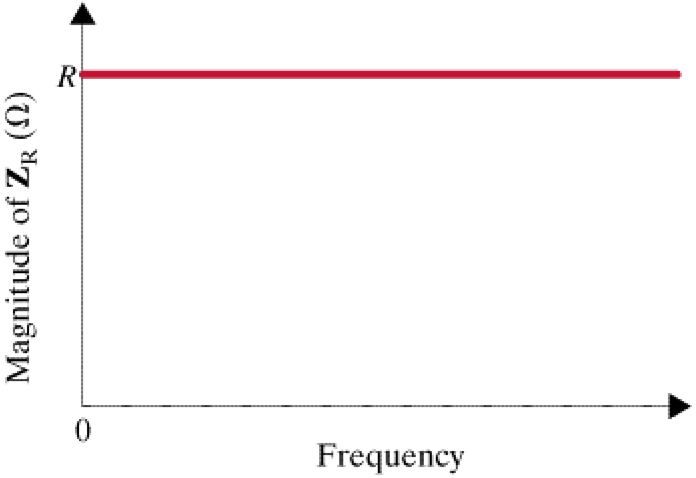
In AC steady state analysis the frequency is assumed constant (e.g., 60Hz). Here we consider the frequency as a variable and examine how the performance varies with the frequency.

Variation in impedance of basic components

Resistor



$$Z_R = R = R\angle 0^\circ$$



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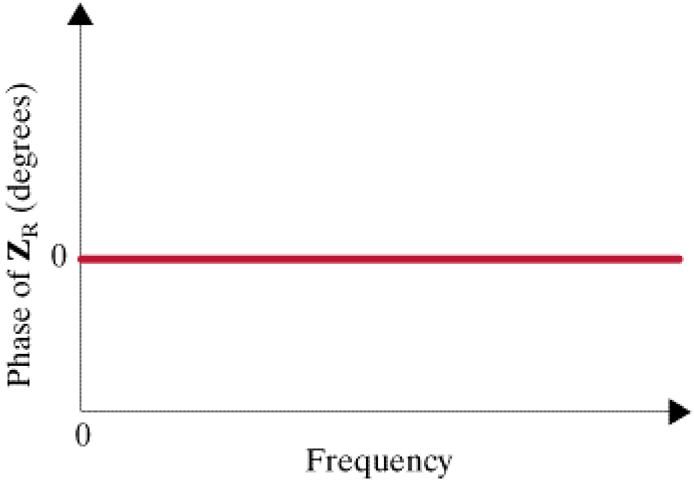

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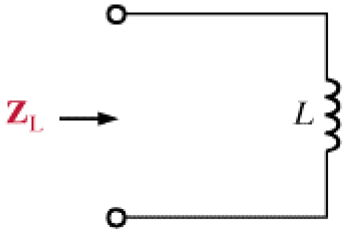
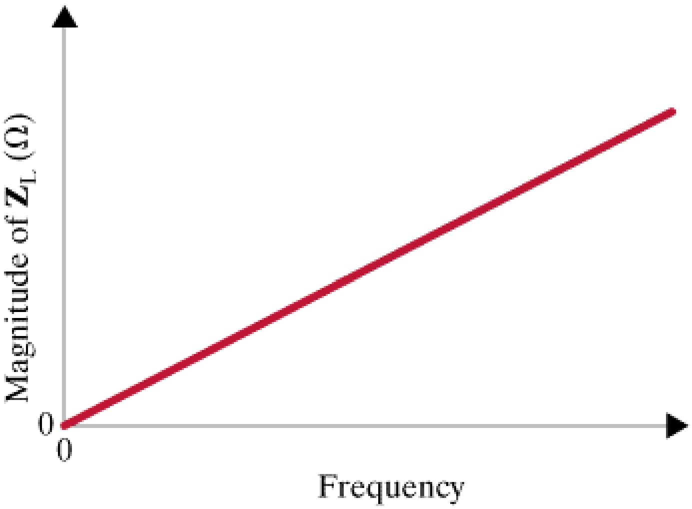
Inductor



$$Z_L = j\omega L = \omega L\angle 90^\circ$$

(a)



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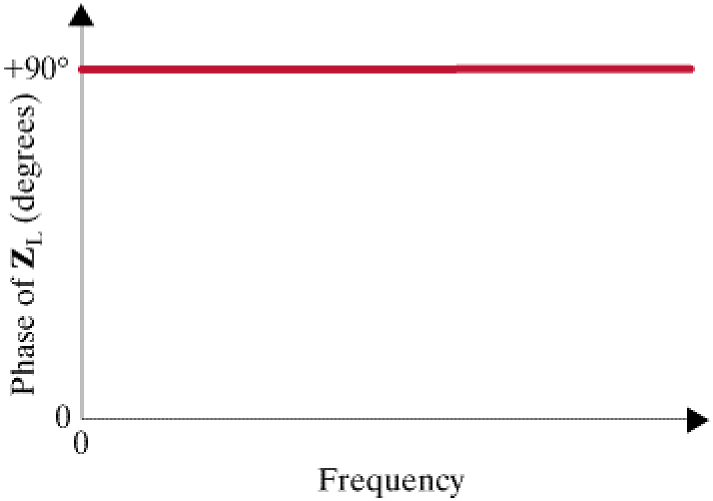

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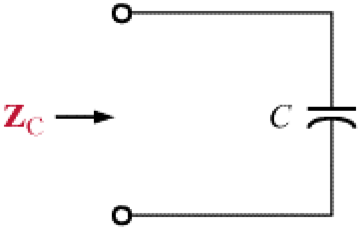
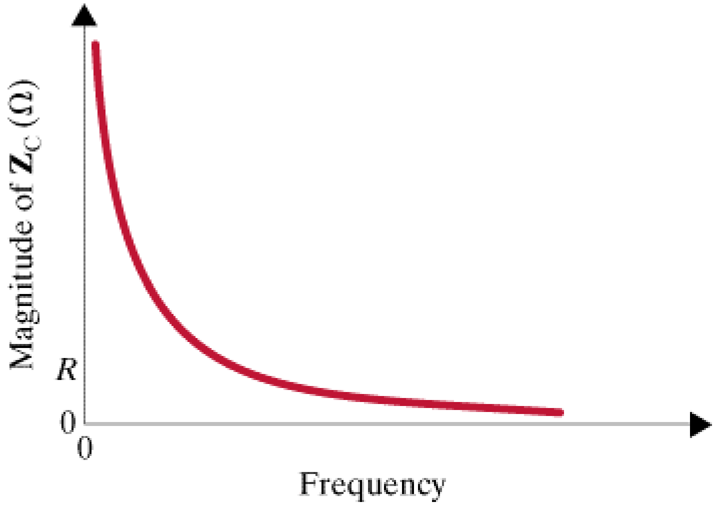
Capacitor



$$Z_c = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90°$$

(a)



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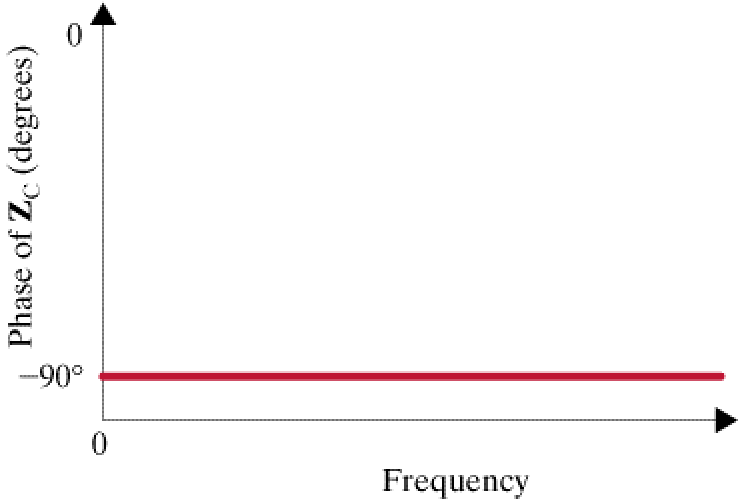

(c)

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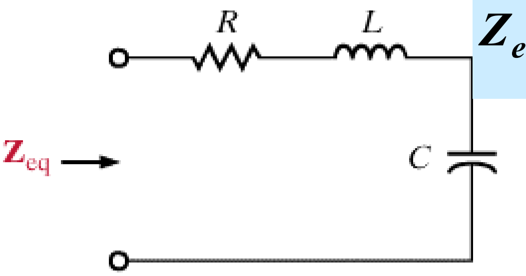
Frequency dependent behavior of series RLC network



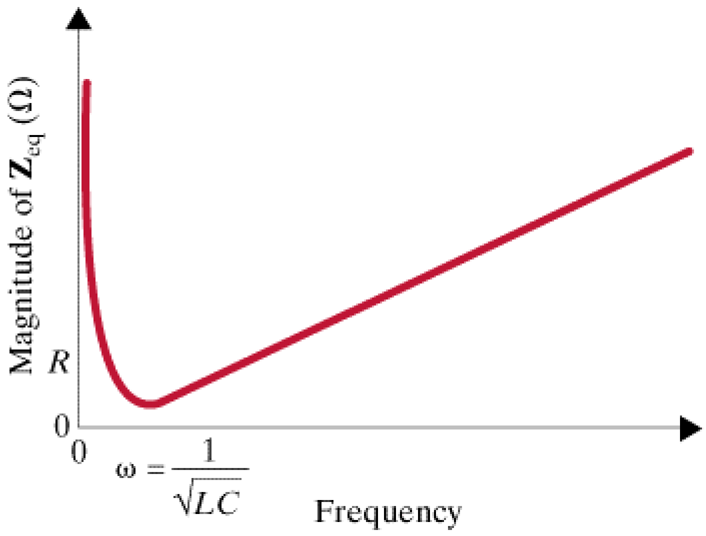
$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C} \times \frac{-j}{-j} = \frac{\omega RC + j(\omega^2 LC - 1)}{\omega C}$$

"Simplification in notation" $j\omega \approx s$

$$Z_{eq}(s) = \frac{s^2 LC + sRC + 1}{sC}$$

$$| Z_{eq} | = \frac{\sqrt{(\omega RC)^2 + (1 - \omega^2 LC)^2}}{\omega C}$$

$$\angle Z_{eq} = \tan^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC}\right)$$



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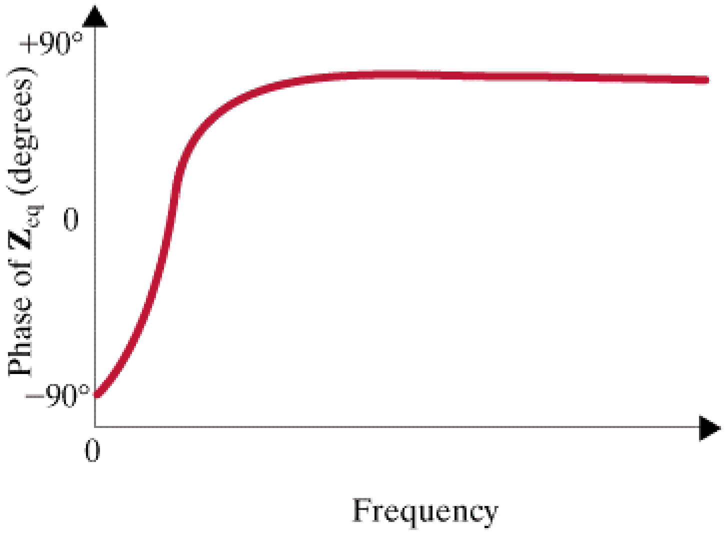

(c)

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Simplified notation for basic components

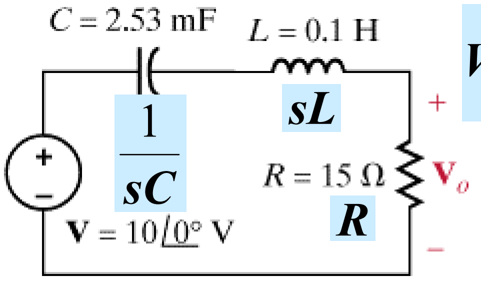
$$Z_R(s) = R,\ \ Z_L(s) = sL,\ \ Z_C = \frac{1}{sC}$$

For all cases seen, and all cases to be studied, the impedance is of the form

$$Z(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}$$

Moreover, if the circuit elements (L,R,C, dependent sources) are real then the expression for any voltage or current will also be a rational function in *s*

LEARNING EXAMPLE



(a)

$$V_o(s) = \frac{R}{R + sL + 1/sC} V_S = \frac{sRC}{s^2 LC + sRC + 1} V_S$$

$$s \approx j\omega$$

$$V_o = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1} V_S$$

$$V_o = \frac{j\omega(15\times 2.53\times 10^{-3})}{(j\omega)^2(0.1\times 2.53\times 10^{-3}) + j\omega(15\times 2.53\times 10^{-3}) + 1} 10\angle 0°$$

MATLAB can be effectively used to compute frequency response characteristics

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USING MATLAB TO COMPUTE MAGNITUDE AND PHASE INFORMATION

$$V_o(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

$$>> num = [a_m, a_{m-1}, \ldots, a_1, a_0];$$
$$>> den = [b_n, b_{n-1}, \ldots, b_1, b_0];$$
$$>>\ freqs(num, den)$$

MATLAB commands required to display magnitude and phase as function of frequency

NOTE: Instead of comma (,) one can use space to separate numbers in the array

EXAMPLE

$$V_o = \frac{\overbrace{j\omega(15\times 2.53\times 10^{-3})}^{a_1}}{(j\omega)^2\underbrace{(0.1\times 2.53\times 10^{-3})}_{b_2} + j\omega\underbrace{(15\times 2.53\times 10^{-3})}_{b_1} + \underbrace{1}_{b_0}}$$

```
» num=[15*2.53*1e-3,0];
» den=[0.1*2.53*1e-3,15*2.53*1e-3,1];
» freqs(num,den)
```

Missing coefficients must be entered as zeros

```
» num=[15*2.53*1e-3 0];
» den=[0.1*2.53*1e-3 15*2.53*1e-3 1];
» freqs(num,den)
```

This sequence will also work. Must be careful not to insert blanks elsewhere

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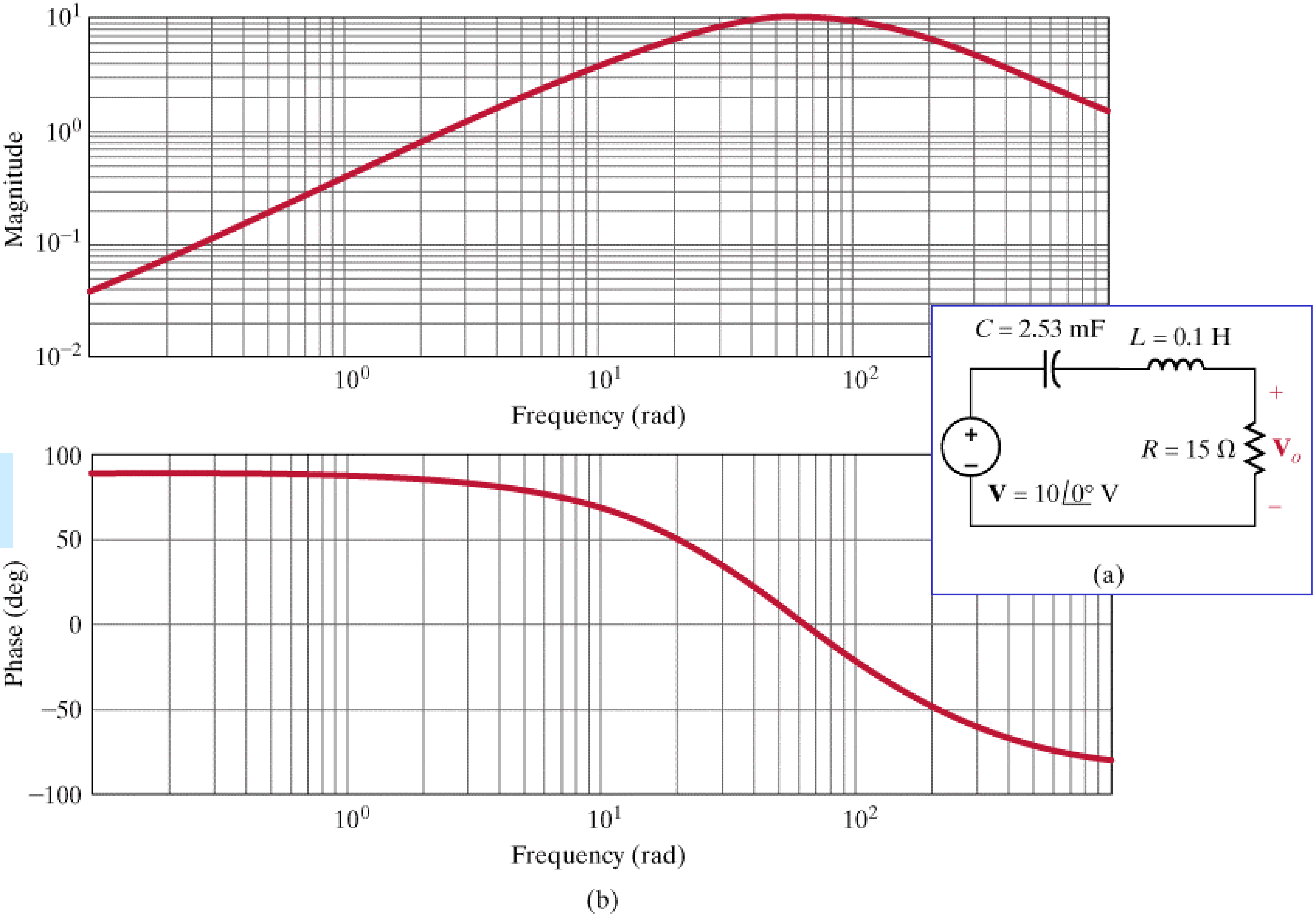
GRAPHIC OUTPUT PRODUCED BY MATLAB

Log-log plot

Semi-log plot



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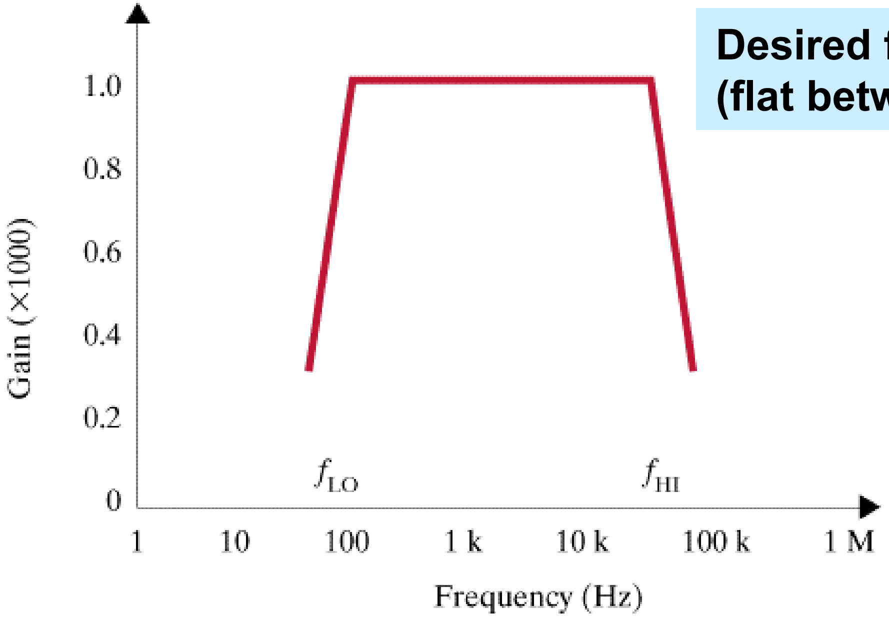
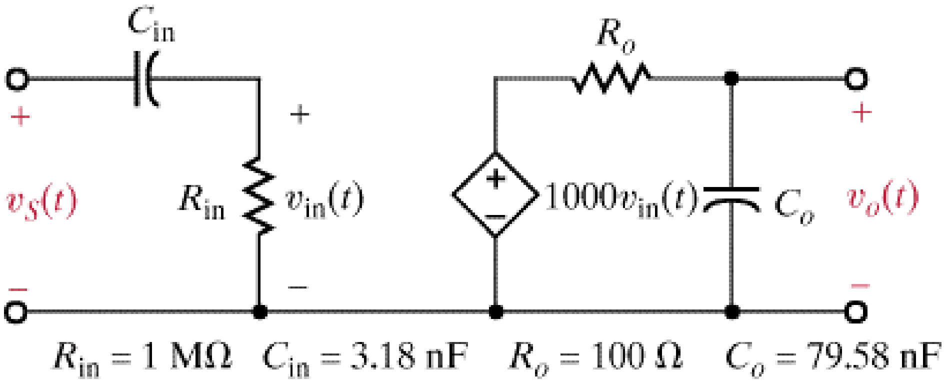
LEARNING EXAMPLE

A possible stereo amplifier



Desired frequency characteristic (flat between 50Hz and 15KHz)

Log frequency scale



(a)

Postulated amplifier

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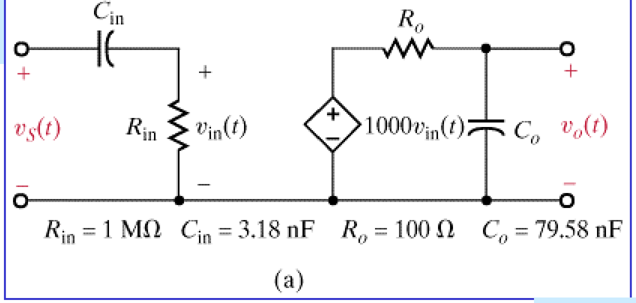
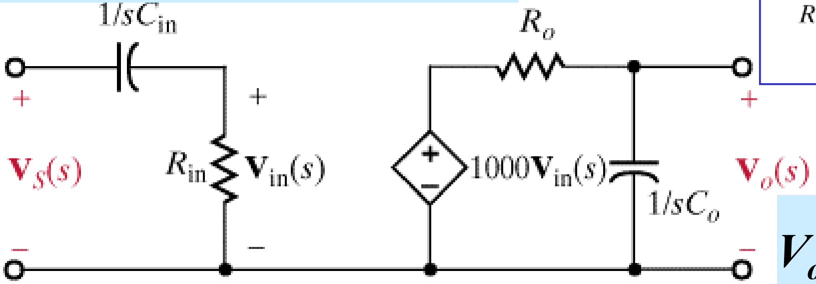
Frequency Analysis of Amplifier



$$V_{in}(s)=\frac{R_{in}}{R_{in}+1/sC_{in}}V_S(s)$$



$$V_o(s)=\frac{1/sC_o}{1/sC_o+R_o}[1000V_{in}]$$

$$G(s)=\frac{V_o(s)}{V_S(s)}=\frac{V_{in}(s)}{V_S(s)}\frac{V_o(s)}{V_{in}(s)}$$

Voltage Gain

Frequency domain equivalent circuit

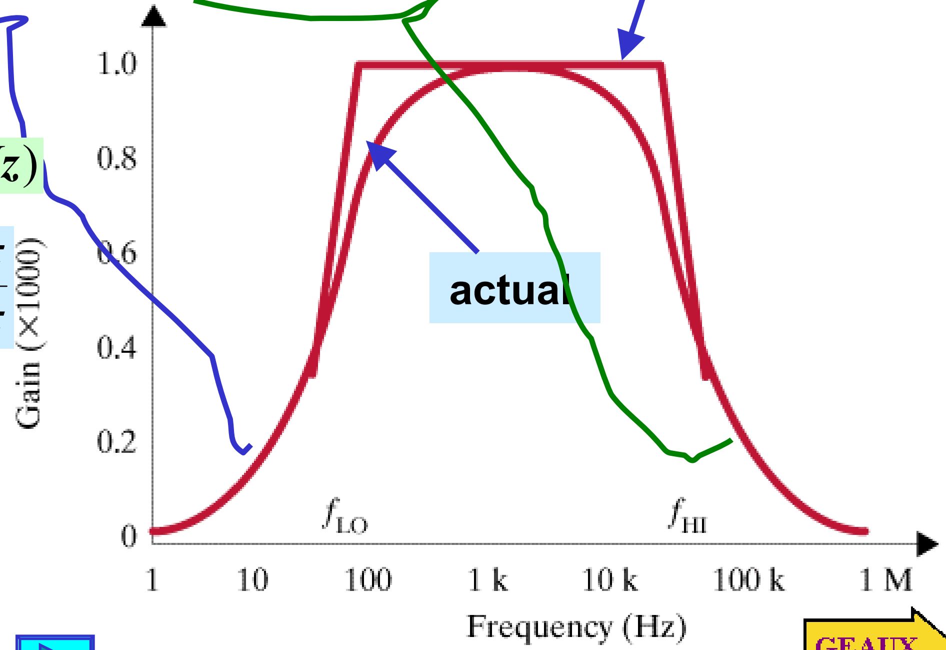
$$G(s)=\left[\frac{sC_{in}R_{in}}{1+sC_{in}R_{in}}\right][1000]\left[\frac{1}{1+sC_oR_o}\right]=\left[\frac{s}{s+100\pi}\right][1000]\left[\frac{40,000\pi}{s+40,000\pi}\right]$$

$$(C_{in}R_{in})^{-1}=\left(3.18\times10^{-9}\times10^{6}\right)^{-1}\approx100\pi\ \ (50Hz)$$

$$(C_oR_o)^{-1}=\left(79.58\times10^{-9}\times100\right)^{-1}\approx40,000\pi(20kHz)$$

$$100\pi<<|s|<<40,000\pi\Rightarrow G(s)\approx\frac{s}{s}[1000]\frac{40,000\pi}{40,000\pi}$$

Frequency dependent behavior is caused by reactive elements



required

actual

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NETWORK FUNCTIONS

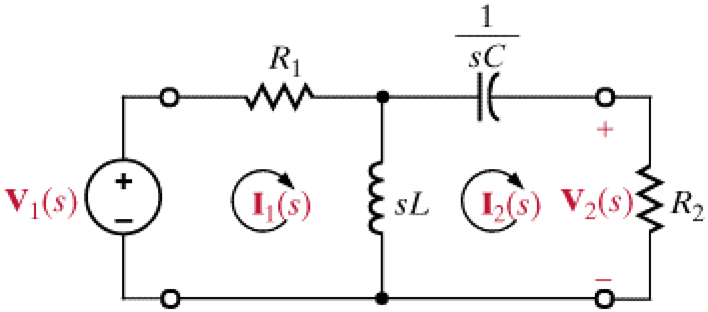
Some nomenclature

When voltages and currents are defined at different terminal pairs we define the ratios as Transfer Functions

INPUT	**OUTPUT**	**TRANSFER FUNCTION**	**SYMBOL**
Voltage	Voltage	Voltage Gain	**Gv(s)**
Current	Voltage	Transimpedance	**Z(s)**
Current	Current	Current Gain	**Gi(s)**
Voltage	Current	Transadmittance	**Y(s)**

If voltage and current are defined at the same terminals we define Driving Point Impedance/Admittance

EXAMPLE



To compute the transfer functions one must solve the circuit. Any valid technique is acceptable

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$

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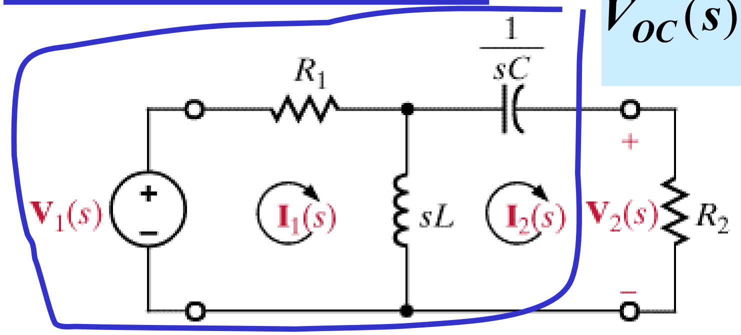
LEARNING EXAMPLE



$$V_{OC}(s) = \frac{sL}{sL + R_1} V_1(s)$$

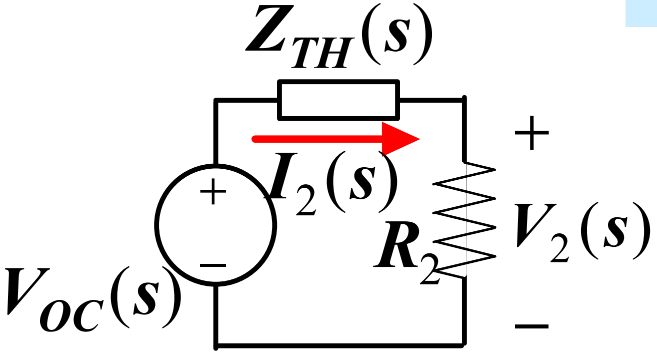
The textbook uses mesh analysis. We will use Thevenin's theorem

$$Z_{TH}(s) = \frac{1}{sC} + R_1 \parallel sL = \frac{1}{sC} + \frac{sLR_1}{sL + R_1}$$

$$Z_{TH}(s) = \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}$$

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$



$$I_2(s) = \frac{V_{OC}(s)}{R_2 + Z_{TH}(s)} = \frac{\frac{sL}{sL + R_1} V_1(s)}{R_2 + \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}} \times \frac{sC(sL + R_1)}{sC(sL + R_1)}$$

$$Y_T(s) = \frac{s^2 LC}{s^2 (R_1 + R_2) LC + s(L + R_1 R_2 C) + R_1}$$

$$G_v(s) = \frac{V_s(s)}{V_1(s)} = \frac{R_2 I_2(s)}{V_1(s)} = R_2 Y_T(s)$$

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POLES AND ZEROS

(More nomenclature)

$$H(s)=\frac{a_m s^m + a_{m-1}s^{m-1}+\ldots+a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}$$

Arbitrary network function

Using the roots, every (monic) polynomial can be expressed as a product of first order terms

$$H(s)=K_0\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$z_1, z_2, \ldots, z_m =$ zeros of the network function

$p_1, p_2, \ldots, p_n =$ poles of the network function

The network function is uniquely determined by its poles and zeros and its value at some other value of s (to compute the gain)

EXAMPLE

zeros : $z_1 = -1,$

poles : $p_1 = -2 + j2, p_2 = -2 - j2$

$H(0) = 1$

$$H(s)=K_0\frac{(s+1)}{(s+2-j2)(s+2+j2)}= K_0\frac{s+1}{s^2+4s+8}$$

$$H(0)=K_0\frac{1}{8}=1 \Rightarrow$$

$$H(s)=8\frac{s+1}{s^2+4s+8}$$

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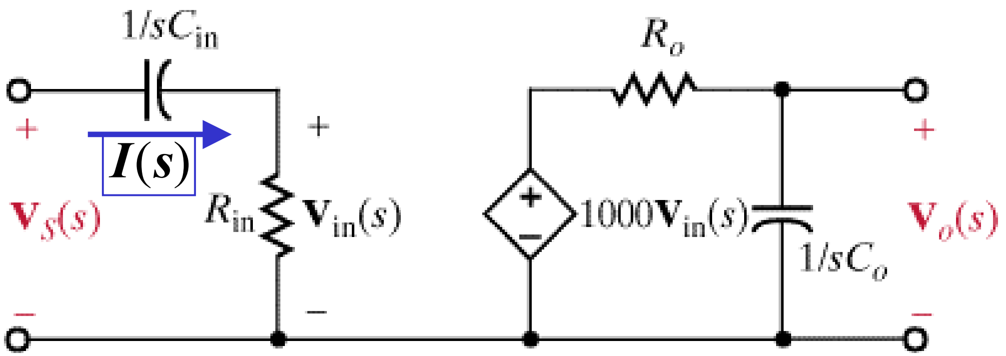
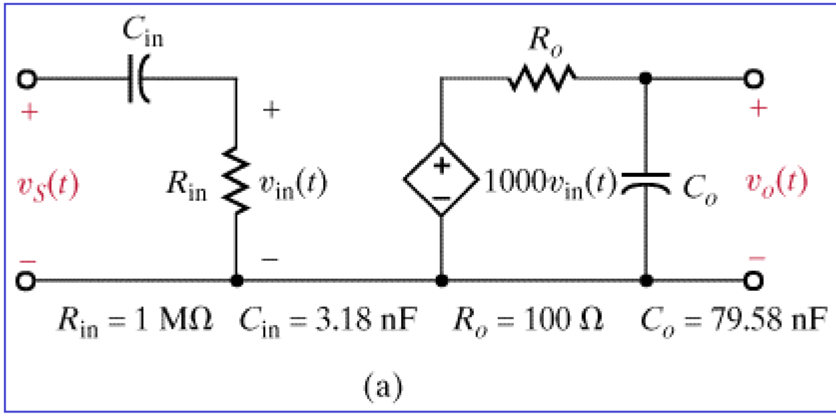
LEARNING EXTENSION

Find the driving point impedance at $V_S(s)$





(a)

$$Z(s)=\frac{V_S(s)}{I(s)}$$

$$\text{KVL}: V_S(s)=R_{in}I(s)+\frac{1}{sC_{in}}I(s)$$

$$Z(s)=R_{in}+\frac{1}{sC_{in}}=\left[1+\frac{100\pi}{s}\right]M\Omega$$

Replace numerical values

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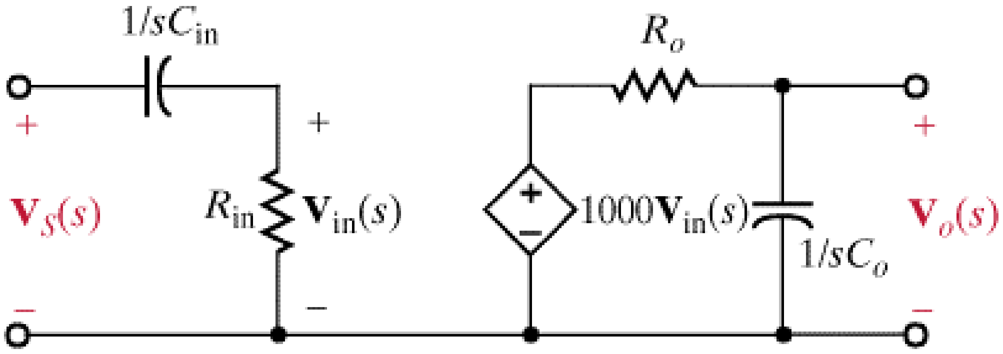
LEARNING EXTENSION

Find the pole and zero locations and the value of K_o for the voltage gain $G(s) = \dfrac{V_o(s)}{V_S(s)}$



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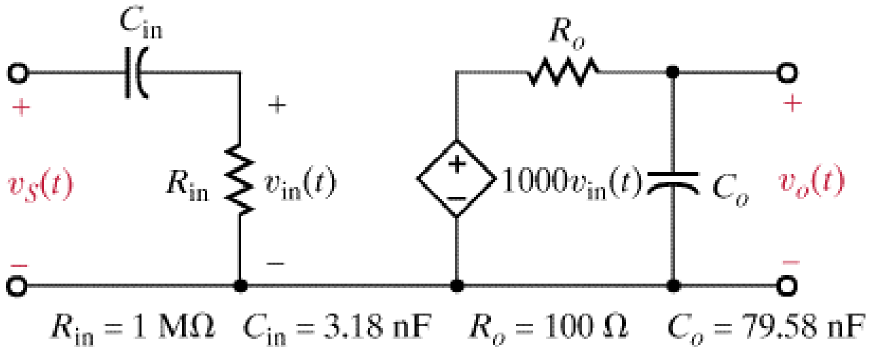


(a)

$$H(s) = K_0 \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

Zeros = roots of numerator
Poles = roots of denominator

For this case the gain was shown to be

$$G(s) = \left[\frac{sC_{in}R_{in}}{1+sC_{in}R_{in}}\right][1000]\left[\frac{1}{1+sC_oR_o}\right] = \left[\frac{s}{s+100\pi}\right][1000]\left[\frac{40{,}000\pi}{s+40{,}000\pi}\right]$$

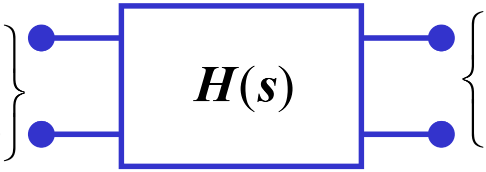
zero: $z_1 = 0$

poles: $p_1 = -50Hz,\ p_2 = -20{,}000Hz$

$K_0 = (4\times10^7)\pi$

Variable Frequency Response

SINUSOIDAL FREQUENCY ANALYSIS



$$\left.\begin{matrix} A_0 e^{j(\omega t+\theta)} \\ B_0\cos(\omega t+\theta) \end{matrix}\right\} \quad H(s) \quad \left\{\begin{matrix} A_0 H(j\omega) e^{j(\omega t+\theta)} \\ B_0 \,|H(j\omega)|\cos\left(\omega t+\theta+\angle H(j\omega)\right) \end{matrix}\right.$$

Circuit represented by network function

To study the behavior of a network as a function of the frequency we analyze the network function $H(j\omega)$ as a function of ω.

Notation

$$M(\omega) = |H(j\omega)|$$

$$\phi(\omega) = \angle H(j\omega)$$

$$H(j\omega) = M(\omega) e^{j\phi(\omega)}$$

Plots of $M(\omega), \phi(\omega)$, as function of ω are generally called magnitude and phase characteristics.

$$\text{BODE PLOTS}\begin{cases} 20\log_{10}(M(\omega)) \\ \phi(\omega) \end{cases} \quad vs\ \log_{10}(\omega)$$

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HISTORY OF THE DECIBEL

Originated as a measure of relative (radio) power

$$P_2\,|_{dB}\ (\text{over } P_1) = 10\log\frac{P_2}{P_1}$$

$$P = I^2R = \frac{V^2}{R} \Rightarrow P_2\,|_{dB}\ (\text{over } P_1) = 10\log\frac{V_2^2}{V_1^2} = 10\log\frac{I_2^2}{I_1^2}$$

By extension

$$V\,|_{dB} = 20\log_{10}|V|$$

$$I\,|_{dB} = 20\log_{10}|I|$$

$$G\,|_{dB} = 20\log_{10}|G|$$

Using log scales the frequency characteristics of network functions have simple asymptotic behavior.

The asymptotes can be used as reasonable and efficient approximations

General form of a network function showing basic terms

Poles/zeros at the origin

Frequency independent

$$H(j\omega)=\frac{K_0(j\omega)^{\pm N}(1+j\omega\tau_1)[1+2\varsigma_3(j\omega\tau_3)+(j\omega\tau_3)^2]\ldots}{(1+j\omega\tau_a)[1+2\varsigma_b(j\omega\tau_b)+(j\omega\tau_b)^2]\ldots}$$

First order terms

Quadratic terms for complex conjugate poles/zeros

$$\log(AB)=\log A+\log B$$

$$\log\left(\frac{N}{D}\right)=\log N-\log D$$

$$\begin{aligned}|H(j\omega)|_{dB}=20\log_{10}|H(j\omega)|\ &=20\log_{10}K_0\pm N20\log_{10}|j\omega|\\ &+20\log_{10}|1+j\omega\tau_1|+20\log_{10}|1+2\varsigma_3(j\omega\tau_3)+(j\omega\tau_3)^2|+\ldots\\ &-20\log_{10}|1+j\omega\tau_a|-20\log_{10}|1+2\varsigma_b(j\omega\tau_b)+(j\omega\tau_b)^2|-..\end{aligned}$$

$$\angle z_1z_2=\angle z_1+\angle z_2$$

$$\angle\frac{z_1}{z_2}=\angle z_1-\angle z_2$$

$$\begin{aligned}\angle H(j\omega)&=0\pm N90^\circ\\ &+\tan^{-1}\omega\tau_1+\tan^{-1}\frac{2\varsigma_3\omega\tau_3}{1-(\omega\tau_3)^2}+\ldots\\ &-\tan^{-1}\omega\tau_a-\tan^{-1}\frac{2\varsigma_b\omega\tau_b}{1-(\omega\tau_b)^2}-\ldots\end{aligned}$$

Display each basic term separately and add the results to obtain final answer

Let's examine each basic term

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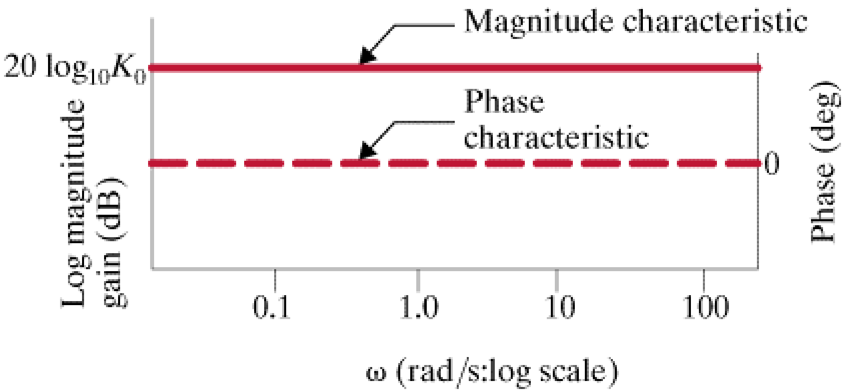
Constant Term



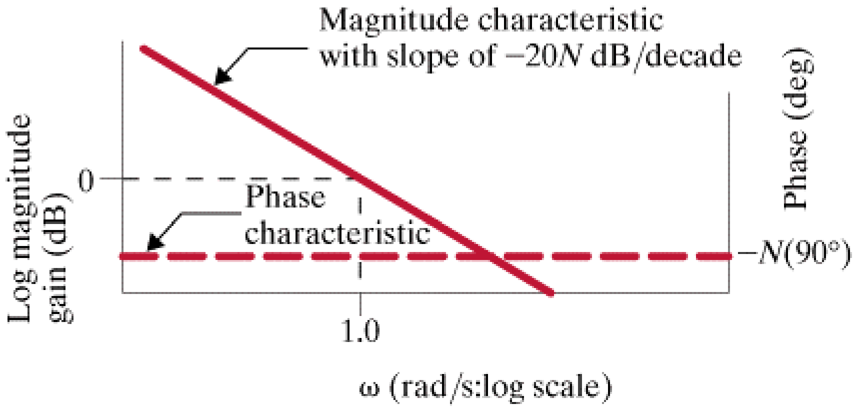
(a)

Poles/Zeros at the origin

$$(j\omega)^{\pm N} \to \begin{cases} |(j\omega)^{\pm N}|_{dB} = \pm N \times 20\log_{10}(\omega) \\ \angle(j\omega)^{\pm N} = \pm N 90^\circ \end{cases}$$

the x - axis is $\log_{10}\omega$

this is a straight line



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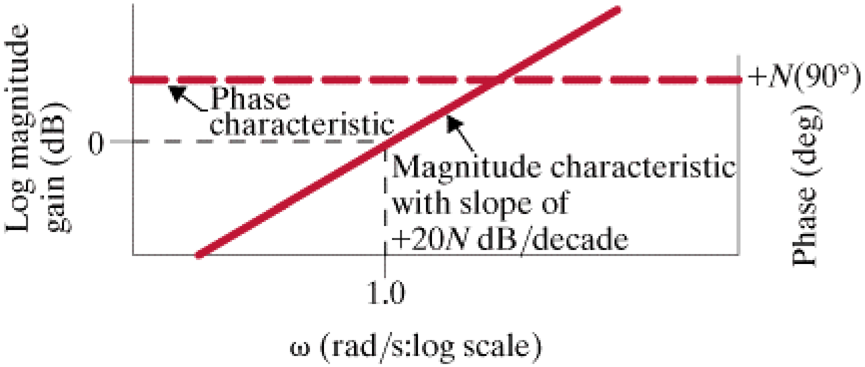

(c)

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Simple pole or zero $1+j\omega\tau$ $\begin{cases} |1+j\omega\tau|_{dB} = 20\log_{10}\sqrt{1+(\omega\tau)^2} \\ \angle(1+j\omega\tau) = \tan^{-1}\omega\tau \end{cases}$

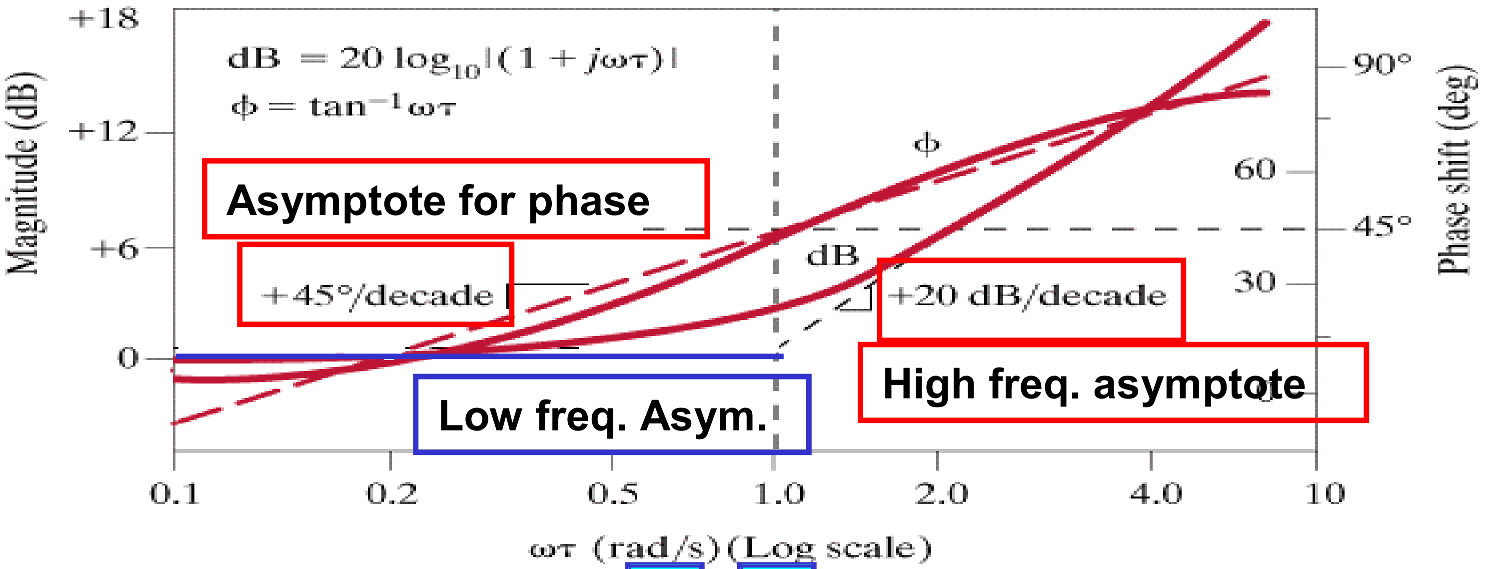
$\omega\tau << 1 \Rightarrow$ $|1+j\omega\tau|_{dB} \approx 0$ low frequency asymptote $\angle(1+j\omega\tau) \approx 0°$

$\omega\tau >> 1 \Rightarrow$ $|1+j\omega\tau|_{dB} \approx 20\log_{10}\omega\tau$ high frequency asymptote (20dB/dec) $\angle(1+j\omega\tau) \approx 90°$

The two asymptotes meet when $\omega\tau = 1$ (corner/break frequency)

Behavior in the neighborhood of the corner

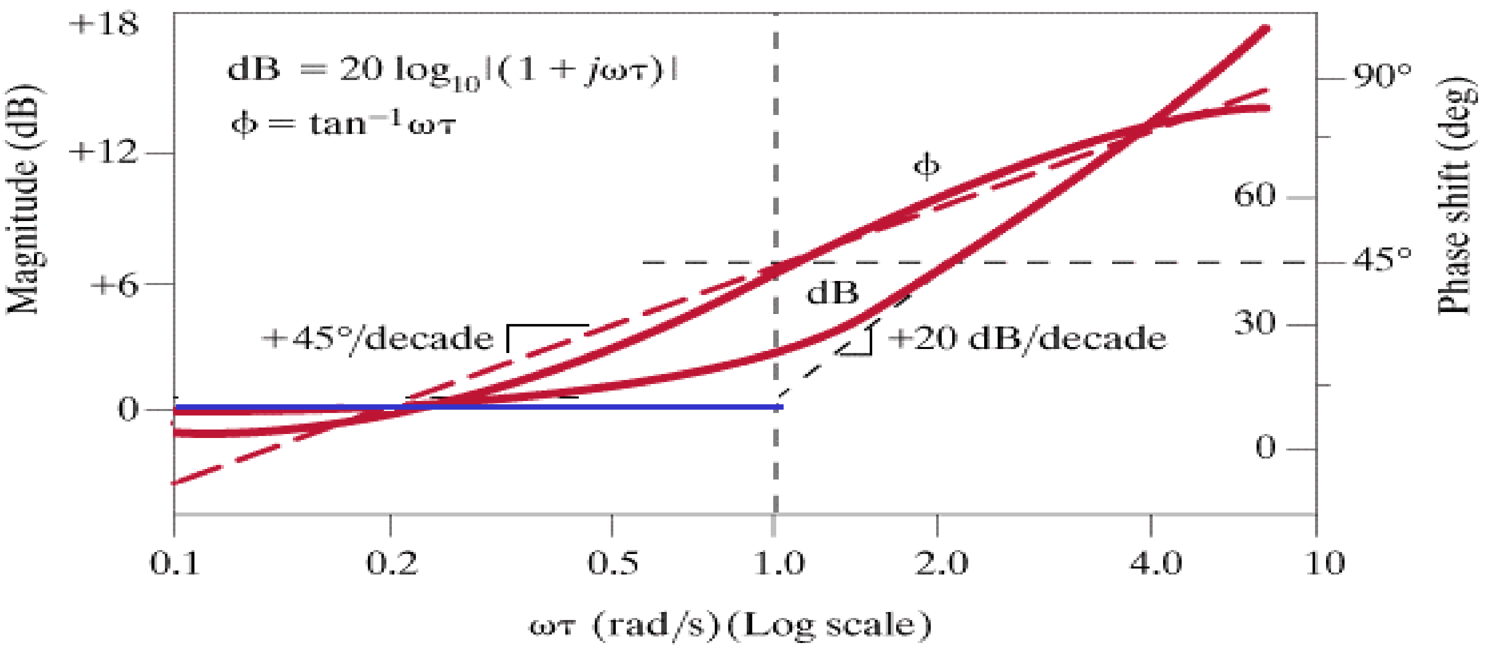
	Frequency	Asymptote	Curve	distance to asymptote	Argument
corner	$\omega\tau = 1$	0dB	3dB	3	45
octave above	$\omega\tau = 2$	6dB	7db	1	63.4
octave below	$\omega\tau = 0.5$	0dB	1dB	1	26.6



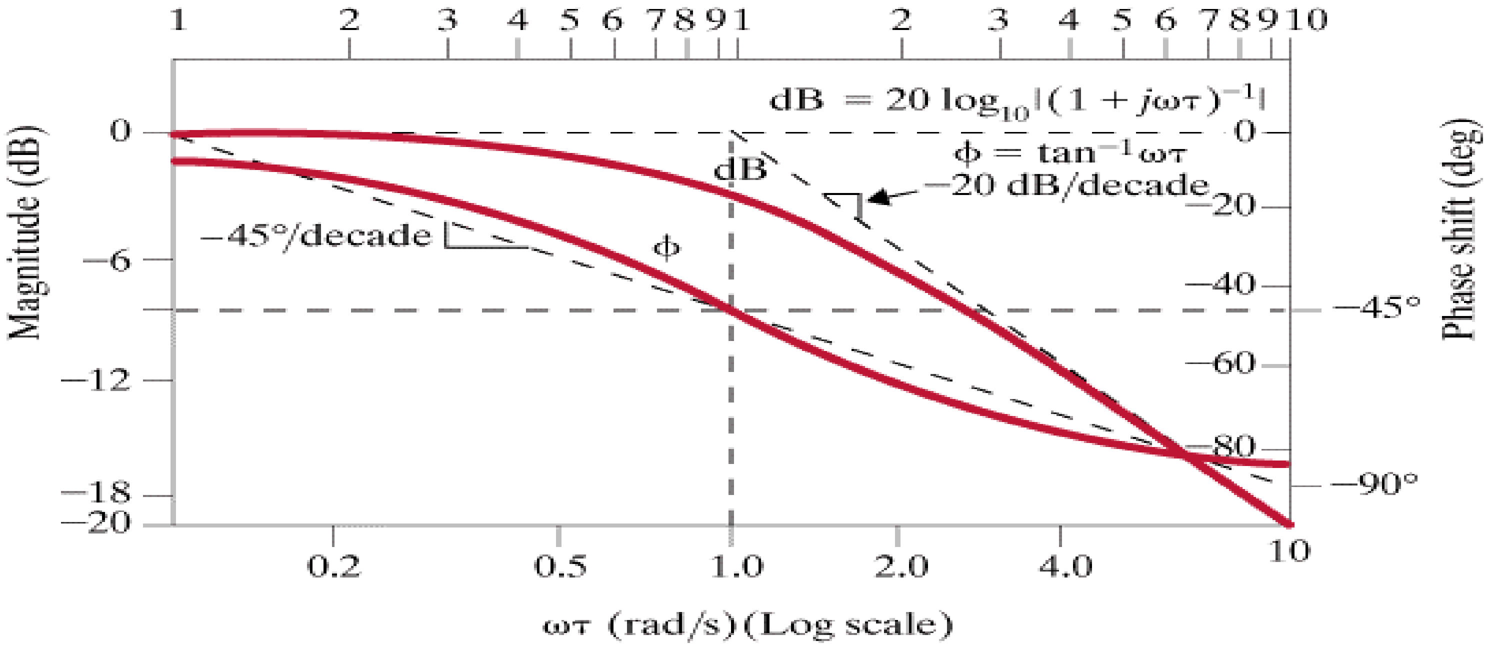
Asymptote for phase

Low freq. Asym.

High freq. asymptote



Simple zero



Simple pole

(a)

GEAUX

Quadratic pole or zero $t_2 = [1+2\varsigma(j\omega\tau)+(j\omega\tau)^2] = [1+2\varsigma(j\omega\tau)-(\omega\tau)^2]$

$|t_2|_{dB} = 20\log_{10}\sqrt{\left(1-(\omega\tau)^2\right)^2+(2\varsigma\omega\tau)^2}$

$\angle t_2 = \tan^{-1}\frac{2\varsigma\omega\tau}{1-(\omega\tau)^2}$

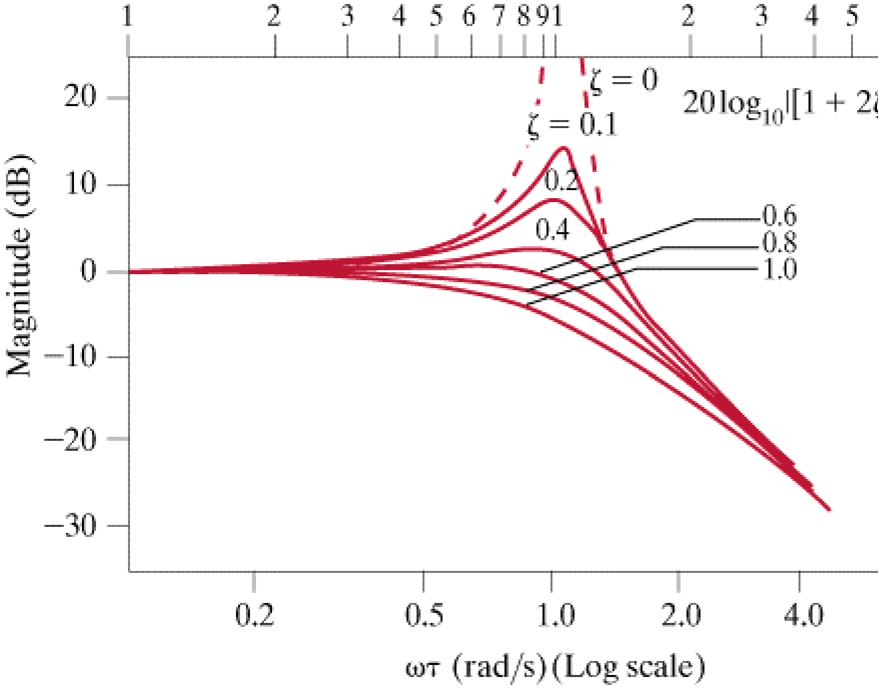
$\omega\tau << 1$ $|t_2|_{dB} \approx 0$ low frequency asymptote $\angle t_2 \approx 0°$

$\omega\tau >> 1$ $|t_2|_{dB} \approx 20\log_{10}(\omega\tau)^2$ high freq. asymptote 40dB/dec $\angle t_2 \approx 180°$

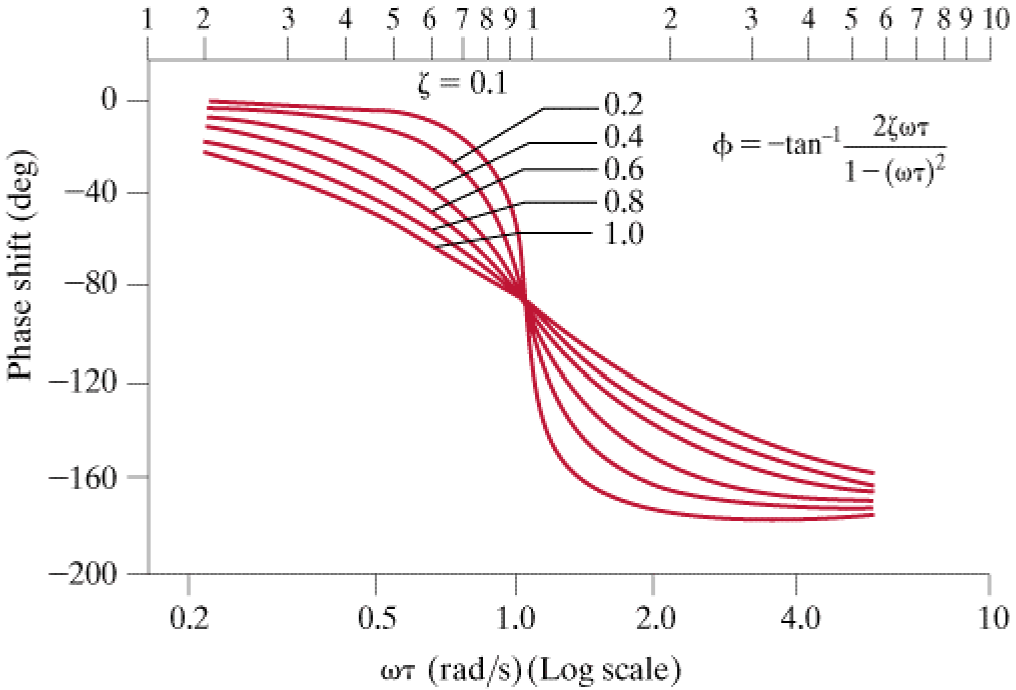
$\omega\tau = 1$ $|t_2|_{dB} = 20\log_{10}(2\varsigma)$ **Corner/break frequency** $\angle t_2 = 90°$

$\omega\tau = \sqrt{1-2\varsigma^2}$ $|t_2|_{dB} = 20\log_{10} 2\varsigma\sqrt{1-\varsigma^2}$ **Resonance frequency** $\angle t_2 = \tan^{-1}\frac{\sqrt{1-2\varsigma^2}}{\varsigma}$ $\varsigma \le \frac{\sqrt{2}}{2}$

These graphs are inverted for a zero



Magnitude for quadratic pole



Phase for quadratic pole

GEAUX

LEARNING EXAMPLE

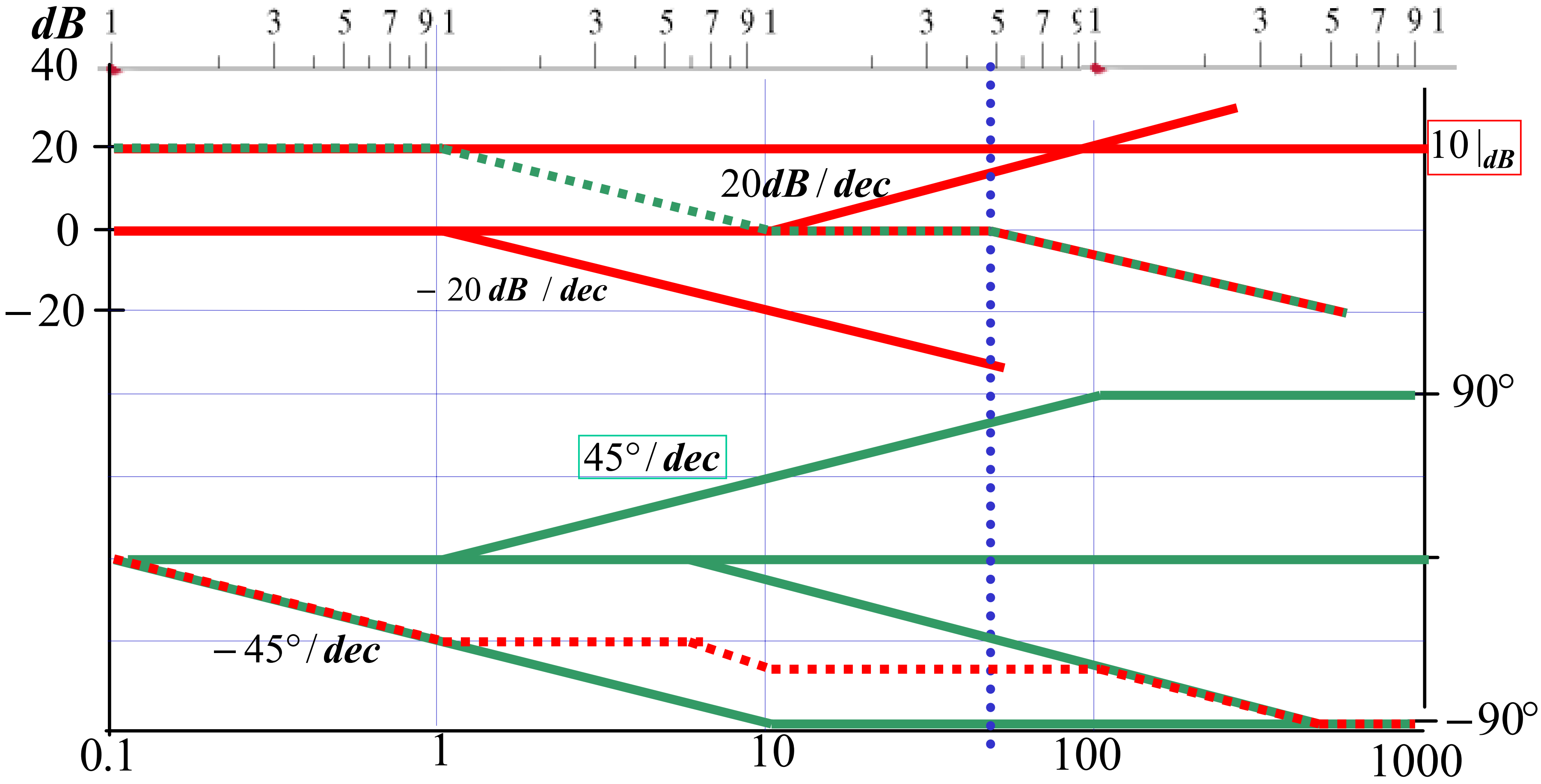
Generate magnitude and phase plots

$$G_v(j\omega) = \frac{\boxed{10}\boxed{(0.1j\omega+1)}}{\boxed{(j\omega+1)}\boxed{(0.02j\omega+1)}}$$

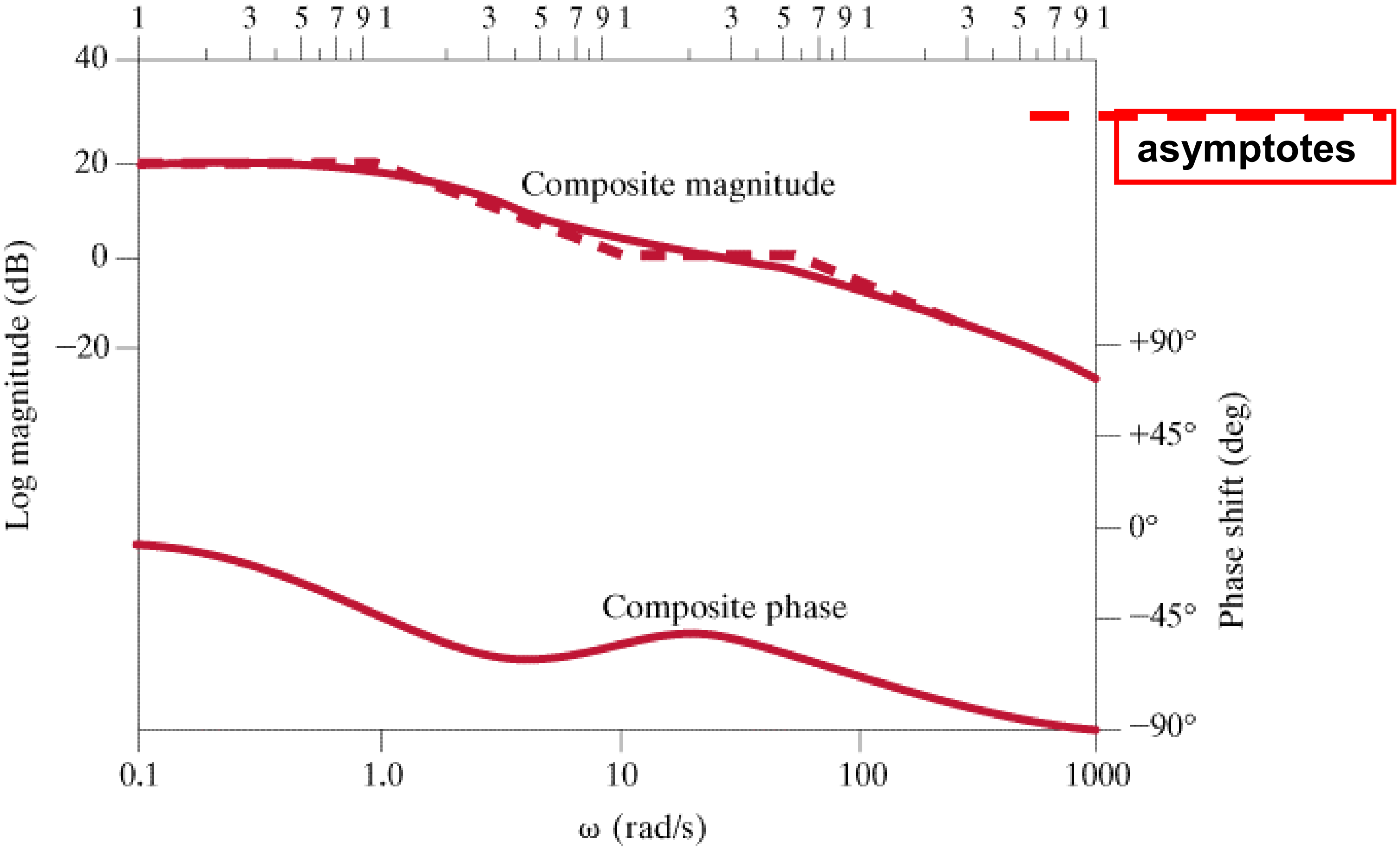
Breaks/corners : 1,10,50

Draw asymptotes for each term

Draw composites



GEAUX


1 3 5 7 9 1 3 5 7 9 1 3 5 7 9 1 3 5 7 9 1
40
20
0
−20
Log magnitude (dB)
asymptotes
Composite magnitude
+90°
+45°
0°
−45°
−90°
Phase shift (deg)
Composite phase
0.1
1.0
10
100
1000
ω (rad/s)

(b)

GEAUX

LEARNING EXAMPLE

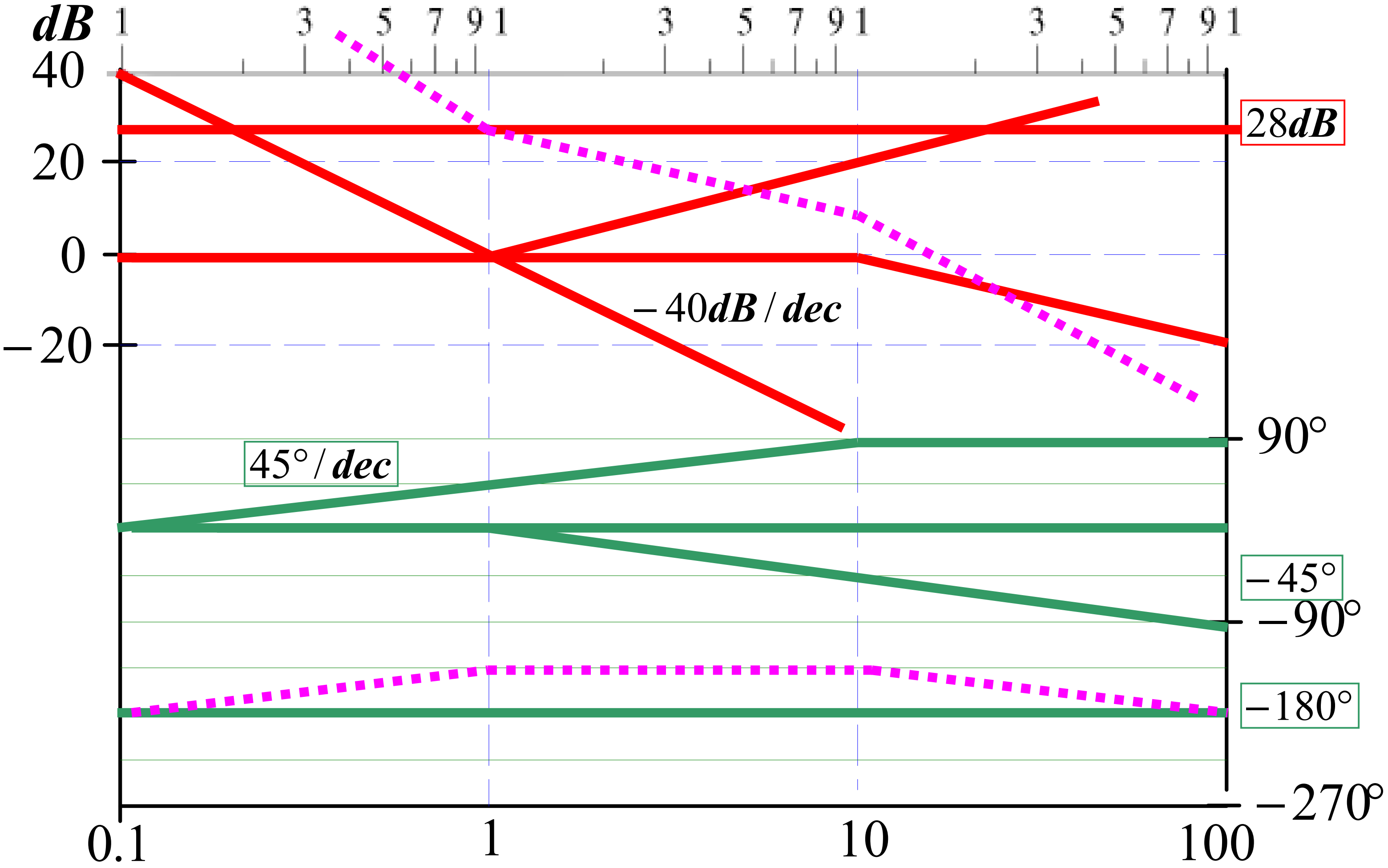
Generate magnitude and phase plots

$$G_v(j\omega) = \frac{25\boxed{(j\omega+1)}}{\boxed{(j\omega)^2}\boxed{(0.1j\omega+1)}}$$

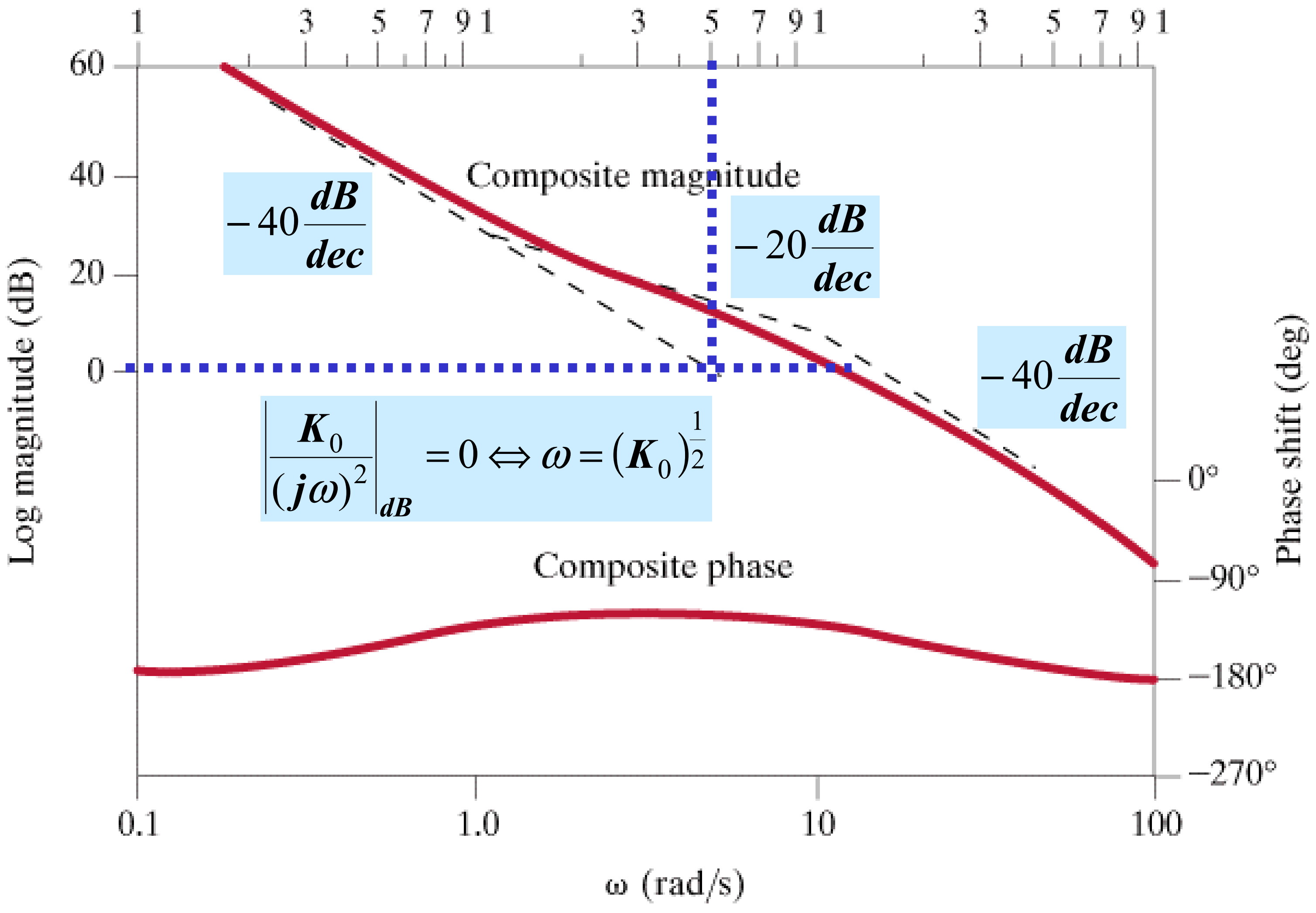
Breaks (corners) : 1, 10

Draw asymptotes for each

Form composites



Final results . . . And an extra hint on poles at the origin



$-40\frac{dB}{dec}$

$-20\frac{dB}{dec}$

$-40\frac{dB}{dec}$

$$\left|\frac{K_0}{(j\omega)^2}\right|_{dB}=0 \Leftrightarrow \omega=\left(K_0\right)^{\frac{1}{2}}$$

(b)

GEAUX

LEARNING EXTENSION

Sketch the magnitude characteristic

$$G(j\omega)=\frac{10^4(j\omega+2)}{(j\omega+10)(j\omega+100)}$$

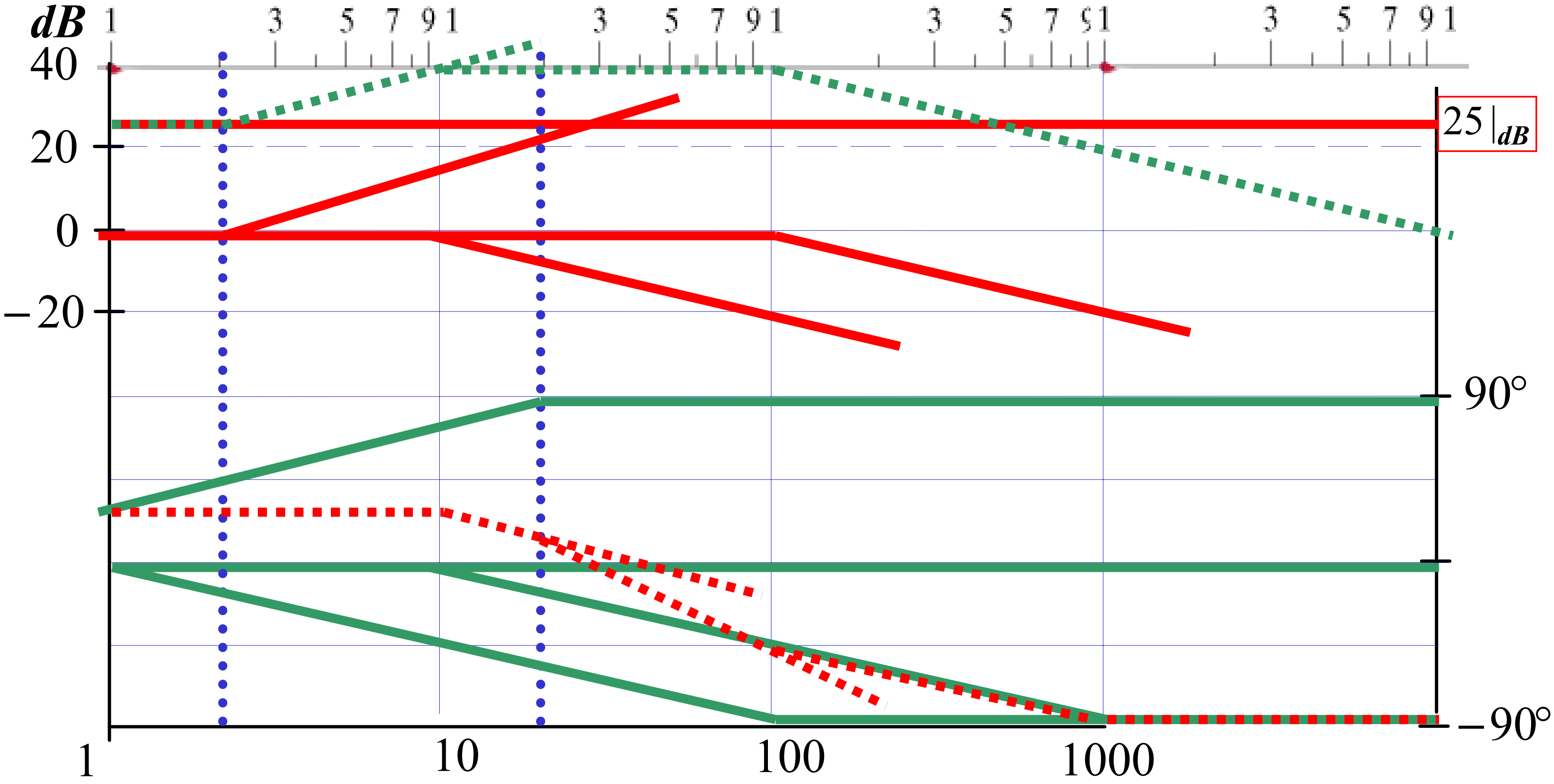
$\text{breaks}: 2, 10, 100$

But the function is NOT in standard form

Put in standard form

$$G(j\omega)=\frac{\boxed{20}\boxed{(j\omega/2+1)}}{\boxed{(j\omega/10+1)}\boxed{(j\omega/100+1)}}$$

We need to show about 4 decades



LEARNING EXTENSION

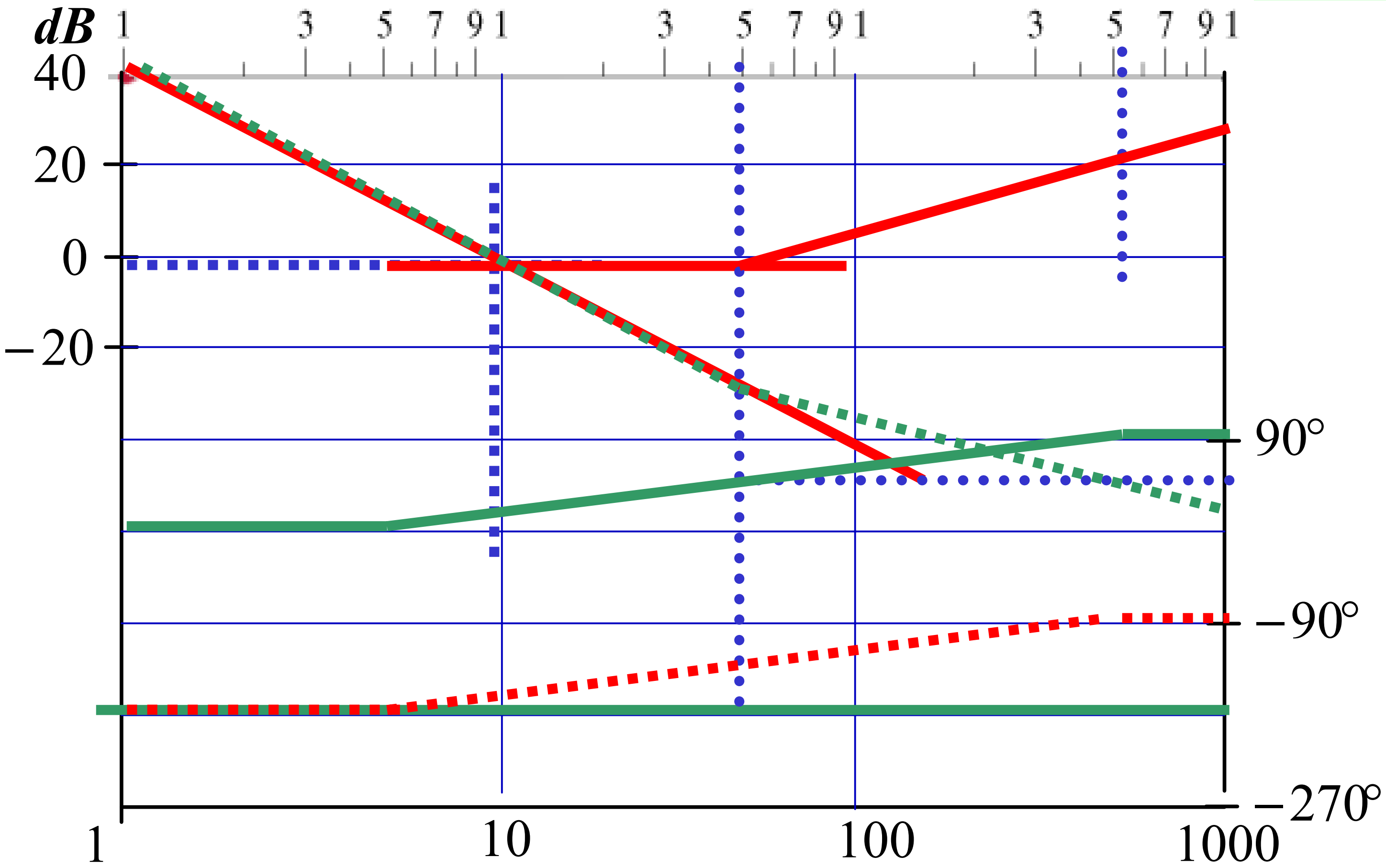
Sketch the magnitude characteristic

$$G(j\omega)=\frac{100(0.02j\omega+1)}{(j\omega)^2}$$

It is in standard form

break at 50

Double pole at the origin



Once each term is drawn we form the composites

LEARNING EXTENSION

Sketch the magnitude characteristic

$$G(j\omega)=\frac{10j\omega}{(j\omega+1)(j\omega+10)}$$

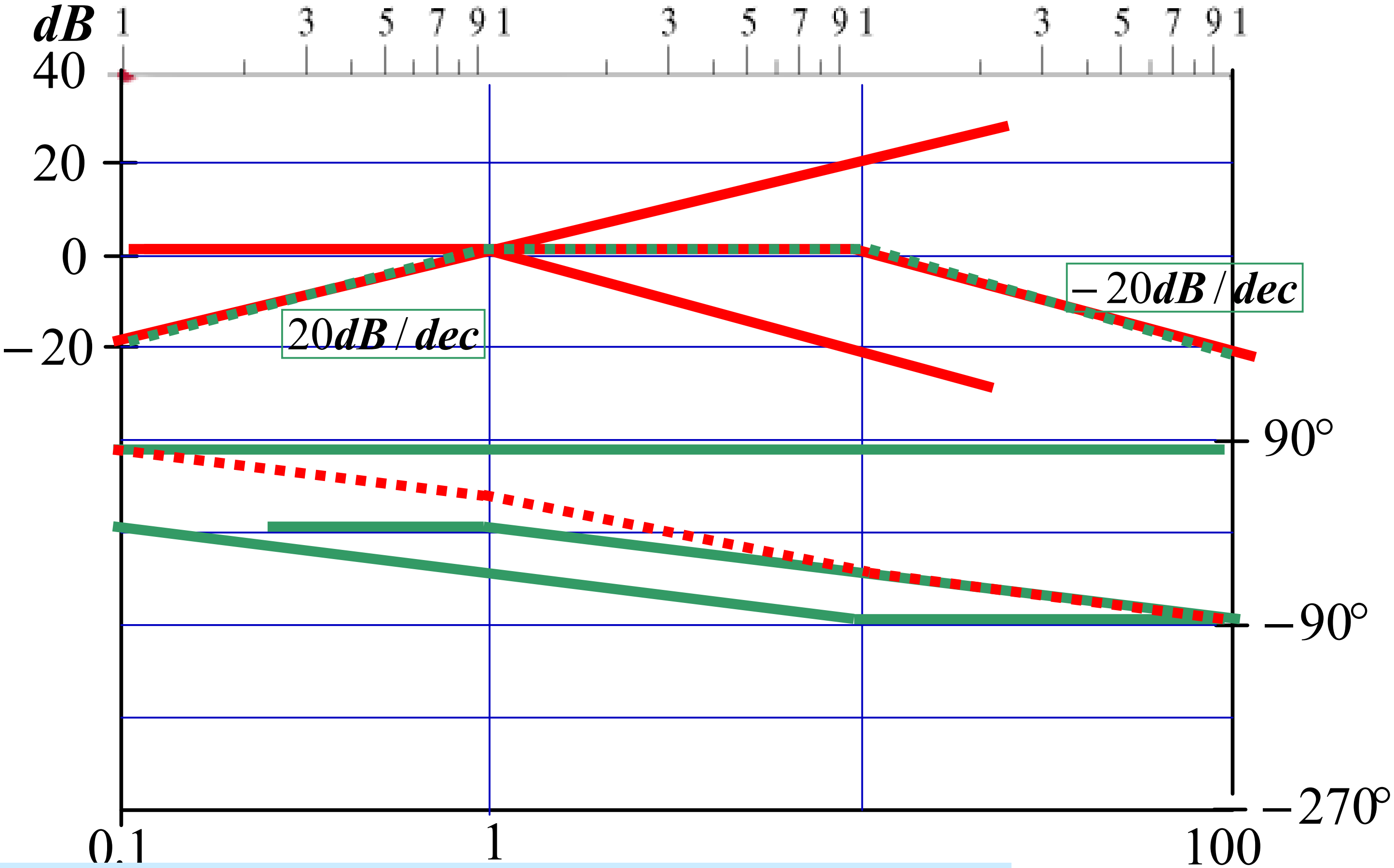
not in standard form

zero at the origin

breaks : 1, 10

Put in standard form

$$G(j\omega)=\frac{\boxed{j\omega}}{\boxed{(j\omega+1)}\boxed{(j\omega/10+1)}}$$



Once each term is drawn we form the composites

GEAUX

LEARNING EXAMPLE — A function with complex conjugate poles

$$t_2 = [1 + 2\varsigma(j\omega\tau) + (j\omega\tau)^2]$$

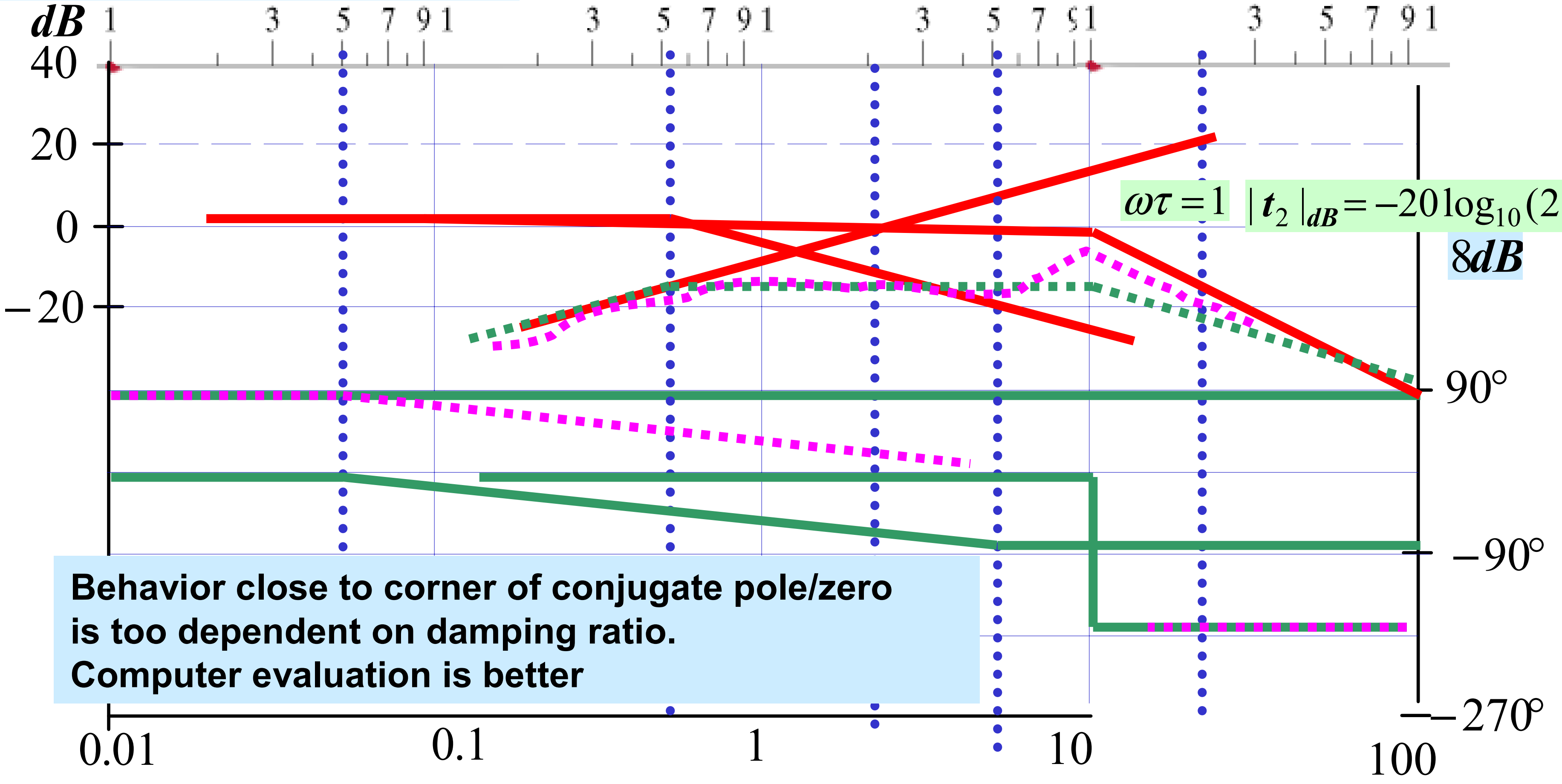
$$G(j\omega) = \frac{25 j\omega}{(j\omega + 0.5)[(j\omega)^2 + 4j\omega + 100]}$$

$$\left.\begin{matrix} 2\varsigma\tau = 1/25 \\ \tau = 0.1 \end{matrix}\right\} \Rightarrow \varsigma = 0.2$$

Put in standard form

$$G(j\omega) = \frac{\boxed{0.5 j\omega}}{\boxed{(j\omega/0.5 + 1)}\boxed{[(j\omega/10)^2 + j\omega/25 + 1]}}$$

Draw composite asymptote



$\omega\tau = 1$ $|t_2|_{dB} = -20\log_{10}(2\varsigma)$

$8dB$

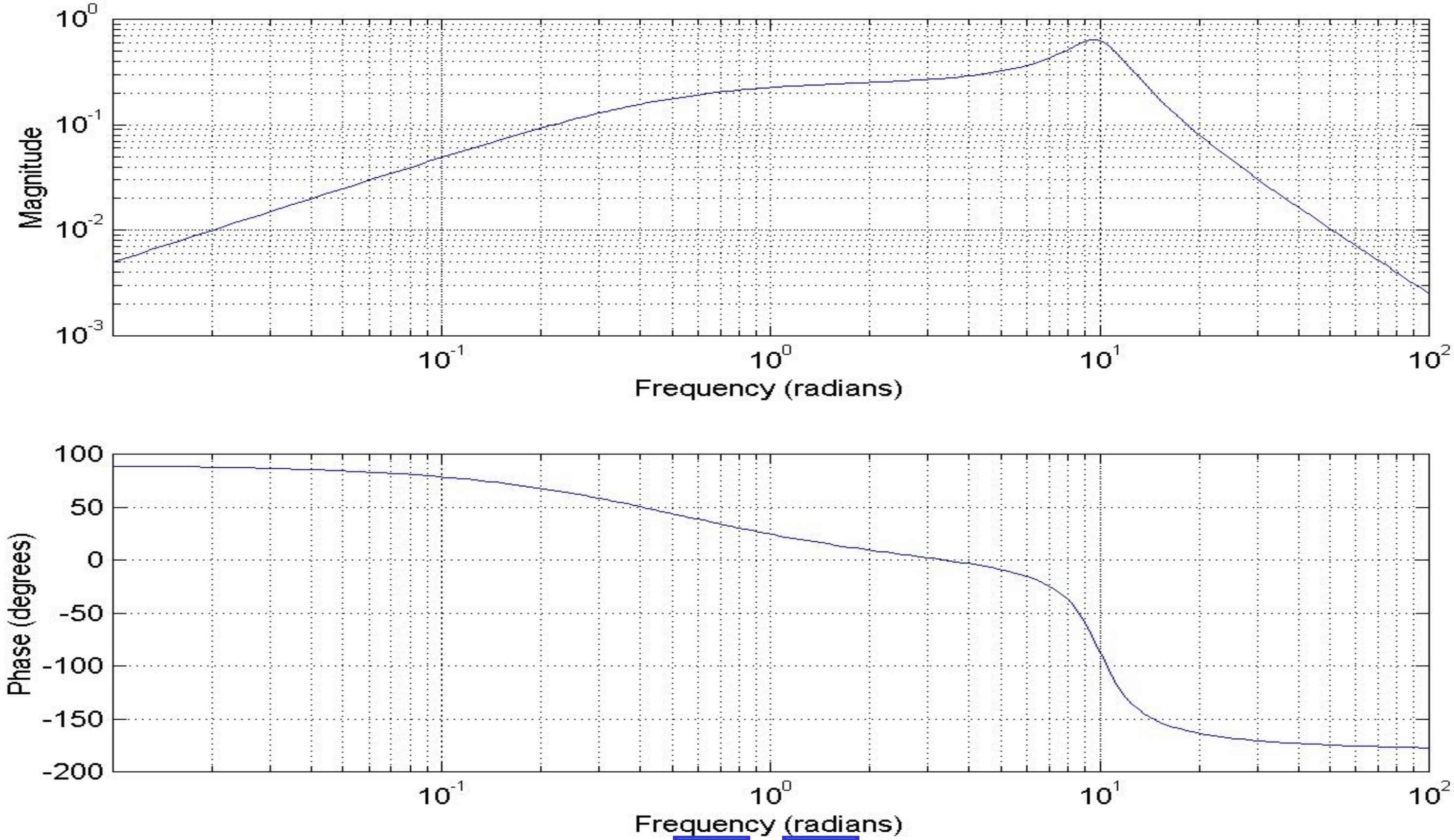
Behavior close to corner of conjugate pole/zero is too dependent on damping ratio.
Computer evaluation is better

GEAUX

Evaluation of frequency response using MATLAB

$$G(j\omega)=\frac{25j\omega}{(j\omega+0.5)\left[(j\omega)^2+4j\omega+100\right]}$$

```
» num=[25,0]; %define numerator polynomial
» den=conv([1,0.5],[1,4,100]) %use CONV for polynomial multiplication
den =
    1.0000    4.5000  102.0000   50.0000
» freqs(num,den)
```



GEAUX

LEARNING EXTENSION

Sketch the magnitude characteristic

$t_2 = [1 + 2\varsigma(j\omega\tau) + (j\omega\tau)^2]$

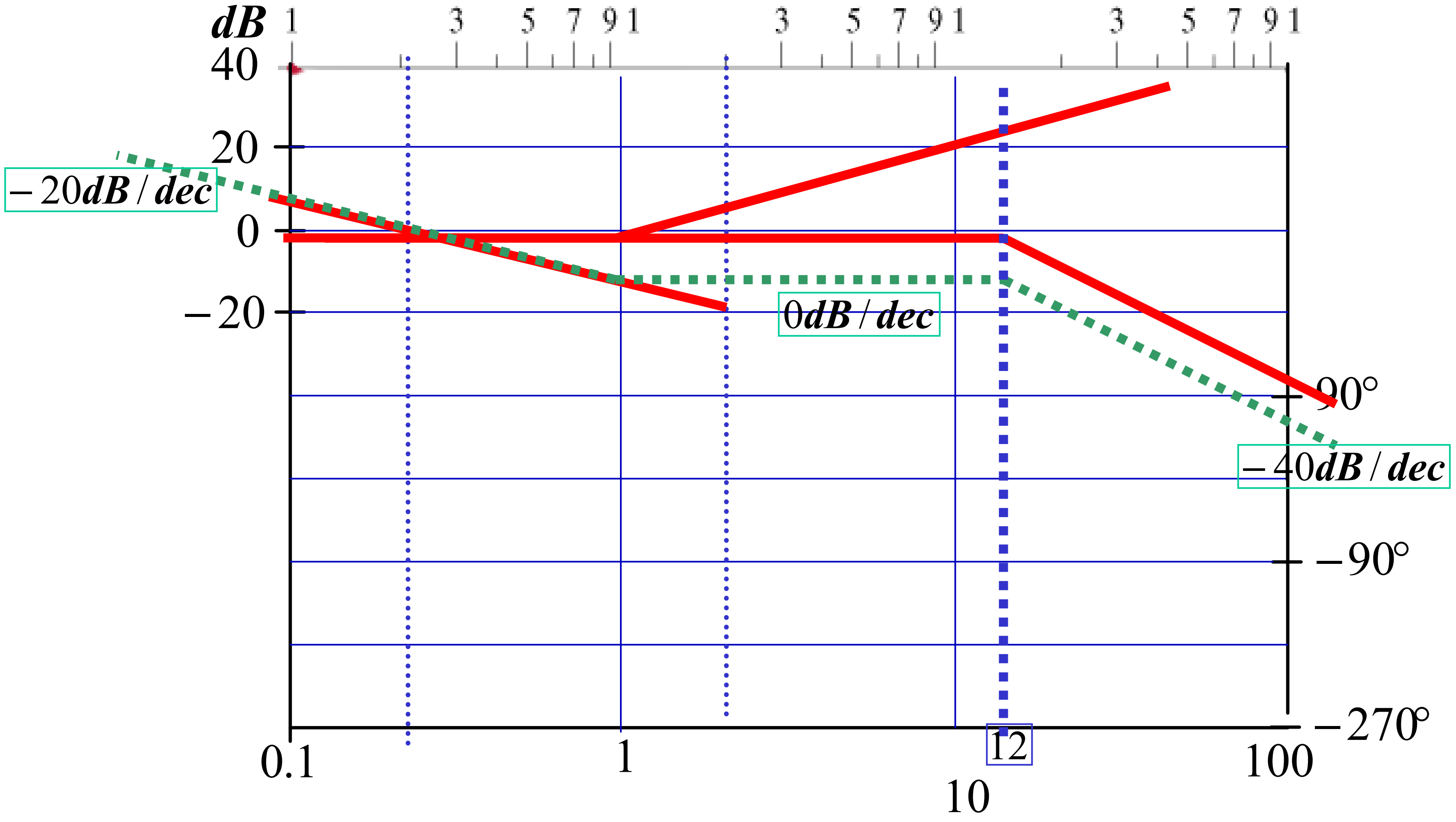
$$G(j\omega) = \frac{0.2\boxed{(j\omega+1)}}{j\omega\boxed{[(j\omega/12)^2 + j\omega/36 + 1]}}$$

$\tau = 1/12$

$2\varsigma\tau = 1/36 \Rightarrow \varsigma = 1/6$

$\omega\tau = 1$ $|t_2|_{dB} = -20\log_{10}(2\varsigma)$

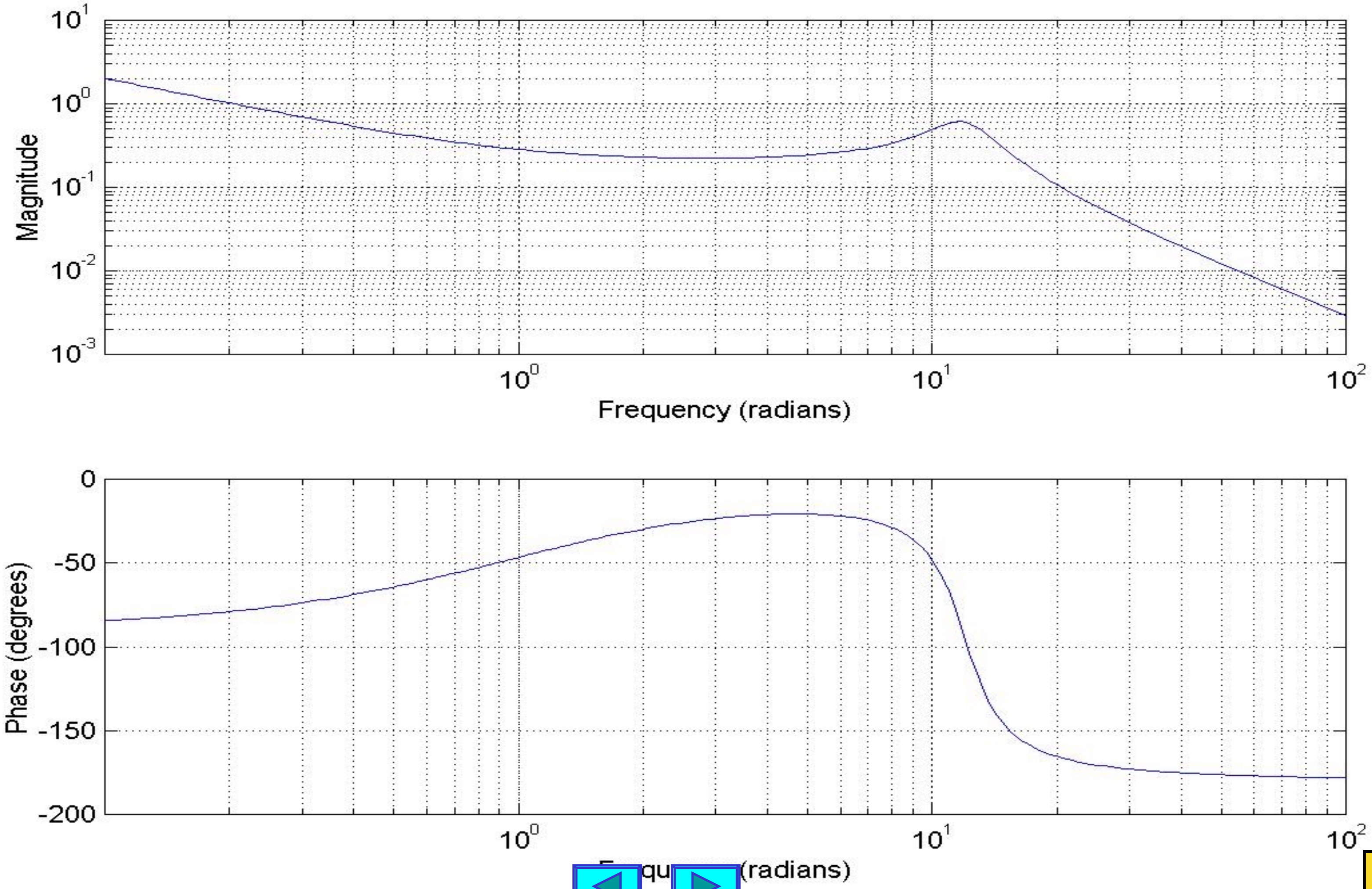
$= 9.5dB$



GEAUX

$$G(j\omega) = \frac{0.2(j\omega+1)}{j\omega[(j\omega/12)^2 + j\omega/36+1]}$$

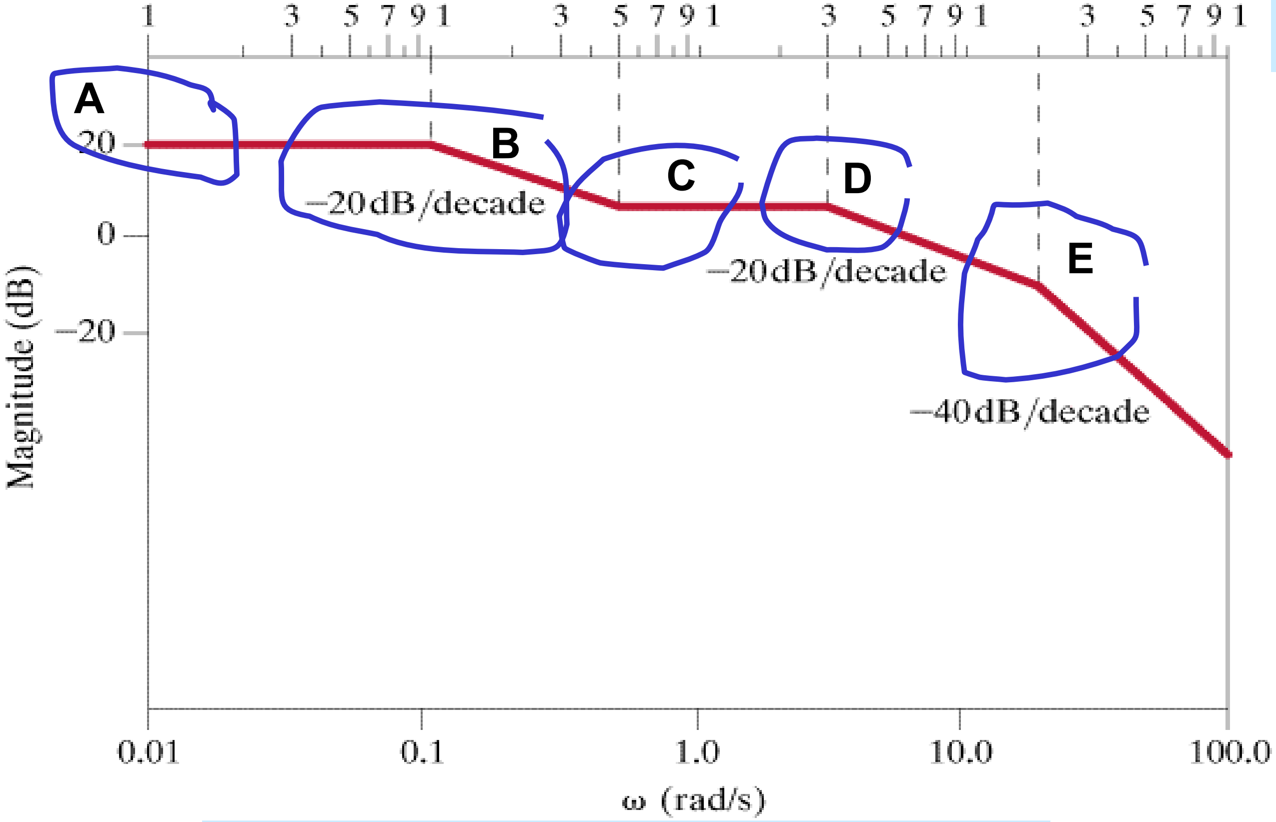
```
» num=0.2*[1,1];
» den=conv([1,0],[1/144,1/36,1]);
» freqs(num,den)
```



GEAUX

DETERMINING THE TRANSFER FUNCTION FROM THE BODE PLOT

This is the inverse problem of determining frequency characteristics.
We will use only the composite asymptotes plot of the magnitude to postulate a transfer function. The slopes will provide information on the order



A. different from 0dB.
There is a constant Ko

$$K_0|_{dB} = 20 \Rightarrow K_0 = 10^{\frac{K_0|_{dB}}{20}}$$

B. Simple pole at 0.1

$$(j\omega/0.1+1)^{-1}$$

C. Simple zero at 0.5

$$(j\omega/0.5+1)$$

D. Simple pole at 3

$$(j\omega/3+1)^{-1}$$

E. Simple pole at 20

$$(j\omega/20+1)^{-1}$$

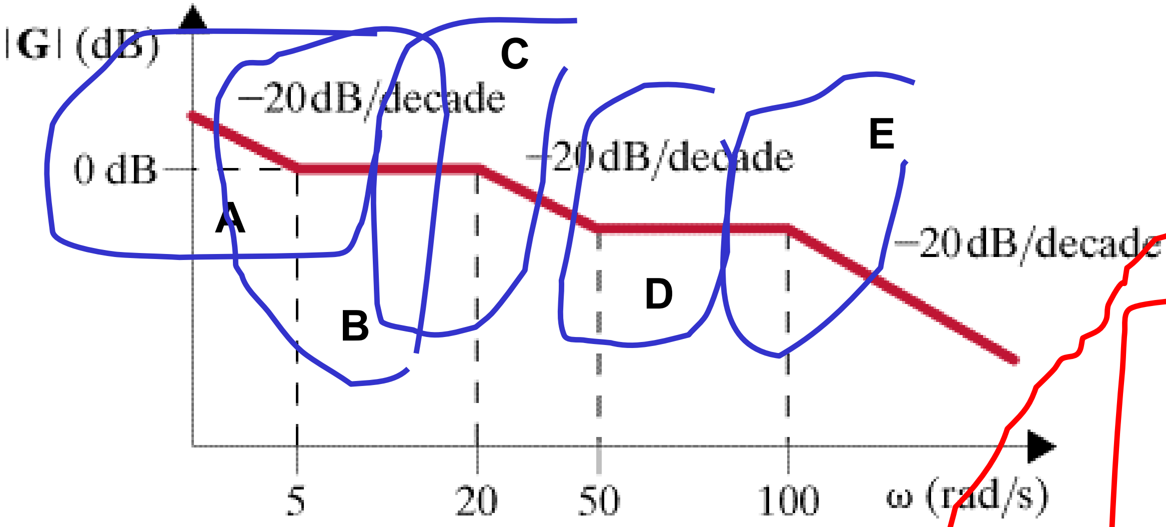
$$G(j\omega) = \frac{10(j\omega/0.5+1)}{(j\omega/0.1+1)(j\omega/3+1)(j\omega/20+1)}$$

If the slope is -40dB we assume double real pole. Unless we are given more data

GEAUX

LEARNING EXTENSION

Determine a transfer function from the composite magnitude asymptotes plot



A. Pole at the origin. Crosses 0dB line at 5

$$\frac{5}{j\omega}$$

B. Zero at 5

C. Pole at 20

D. Zero at 50

E. Pole at 100

$$G(j\omega) = \frac{5(j\omega/5+1)(j\omega/50+1)}{j\omega(j\omega/20+1)(j\omega/100+1)}$$

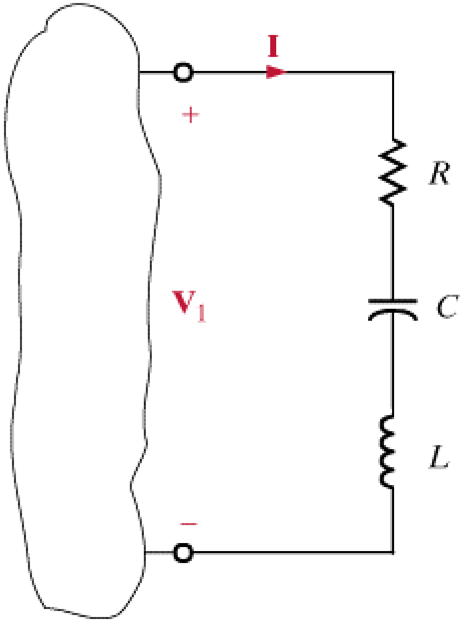
Sinusoidal

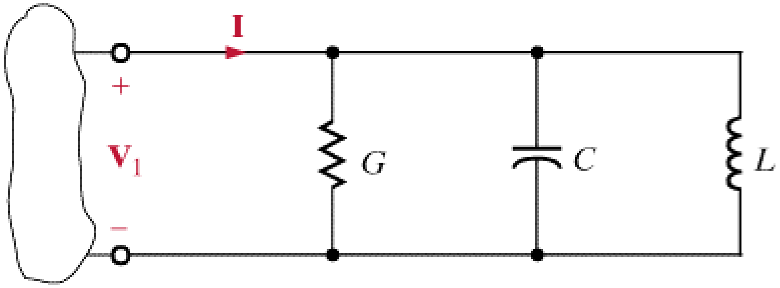
STOP

◀ ▶

RESONANT CIRCUITS

These are circuits with very special frequency characteristics…
And resonance is a very important physical phenomenon





Parallel RLC circuit

$$Y(j\omega) = G + j\omega C + \frac{1}{j\omega L}$$

Series RLC circuit

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

The reactance of each circuit is zero when

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

The frequency at which the circuit becomes purely resistive is called the resonance frequency

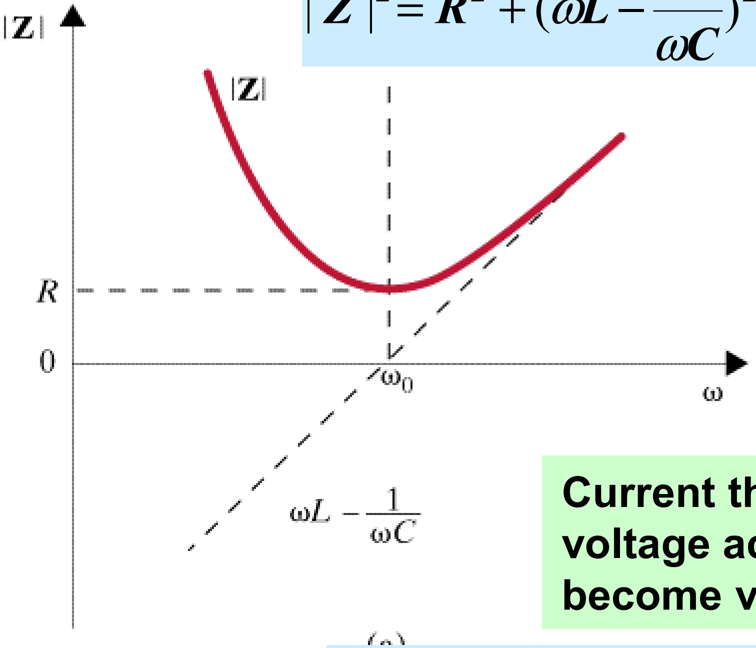
GEAUX

Properties of resonant circuits

At resonance the impedance/admittance is minimal

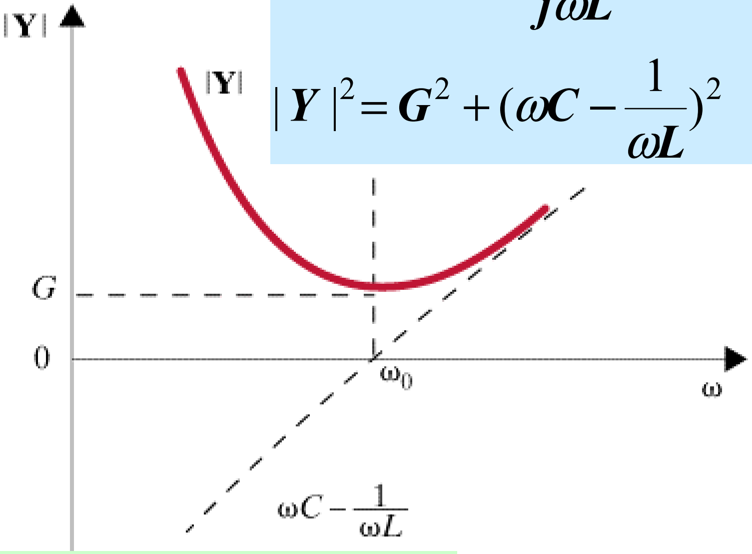
$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$



$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

$$|Y|^2 = G^2 + (\omega C - \frac{1}{\omega L})^2$$



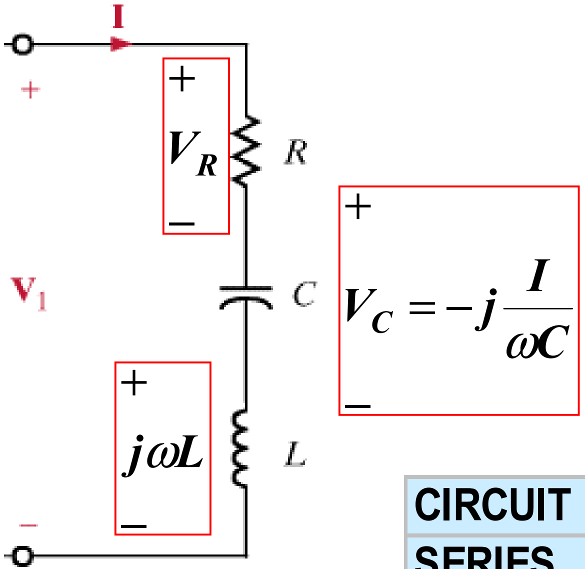
Current through the serial circuit/ voltage across the parallel circuit can become very large (if resistance is small)

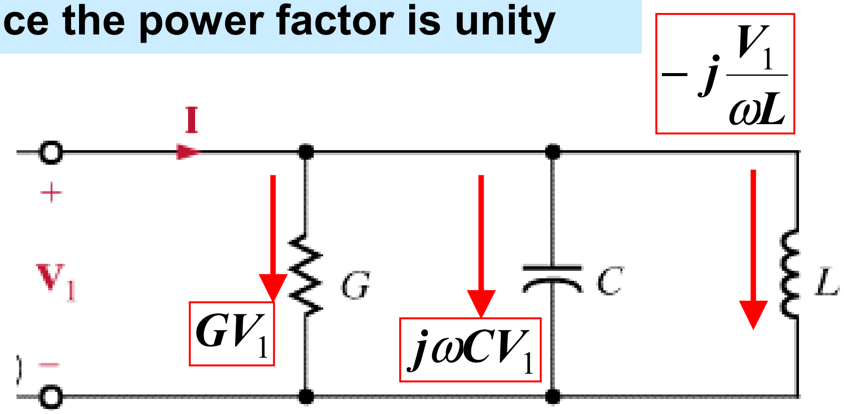
$$\text{Quality Factor}: \; Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Given the similarities between series and parallel resonant circuits, we will focus on serial circuits

Properties of resonant circuits

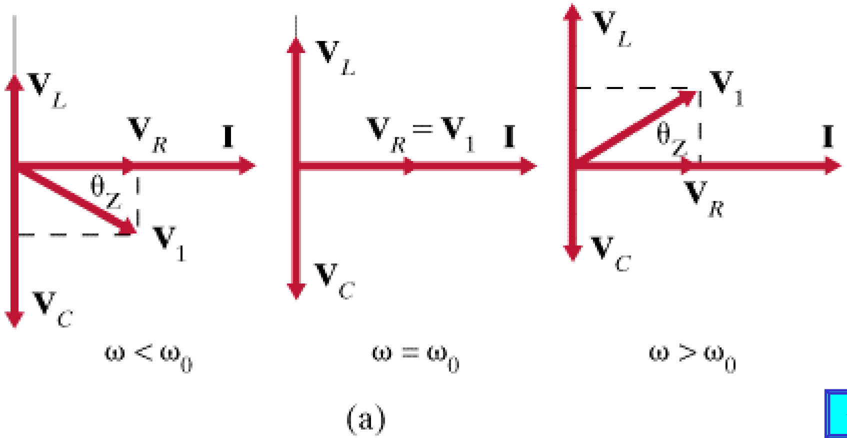
At resonance the power factor is unity





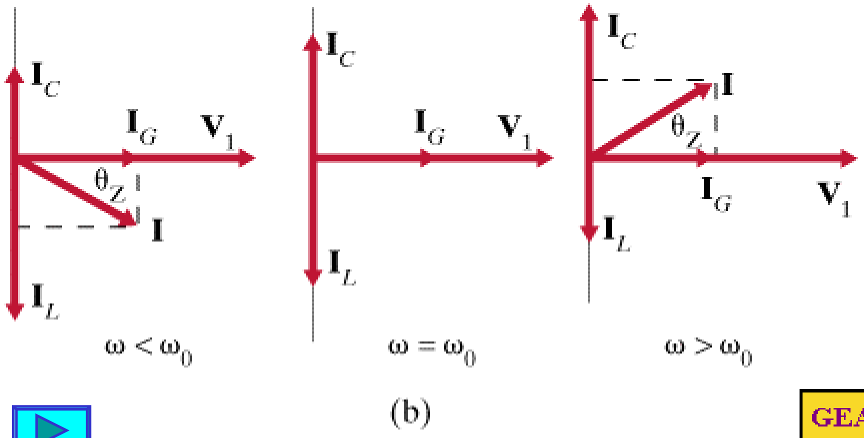
CIRCUIT	BELOW RESONANCE	ABOVE RESONANCE
SERIES	CAPACITIVE	INDUCTIVE
PARALLEL	INDUCTIVE	CAPACITIVE

Phasor diagram for series circuit



(a)

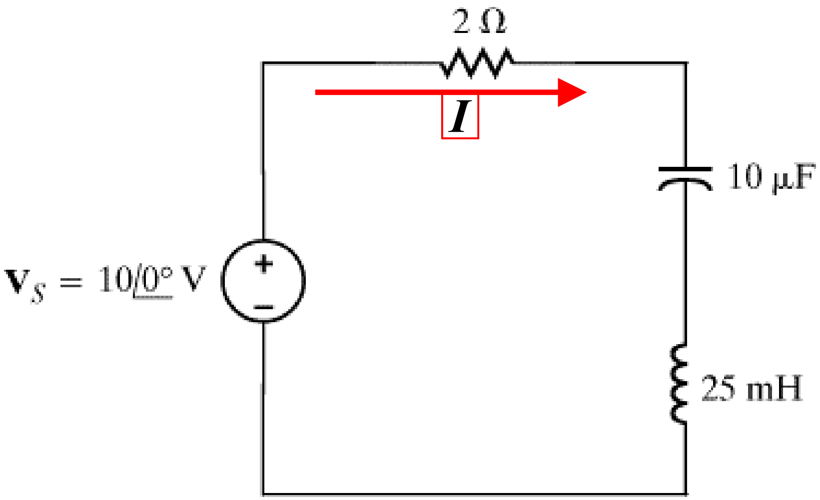
Phasor diagram for parallel circuit



(b)

LEARNING EXAMPLE

Determine the resonant frequency, the voltage across each element at resonance and the value of the quality factor



$$\frac{1}{\omega_0 C} = \omega_0 L = 50\Omega$$

$$V_C = \frac{1}{j\omega_0 C} I = -j50\times 5 = 250\angle -90°$$

$$Q = \frac{\omega_0 L}{R} = \frac{50}{2} = 25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(25\times 10^{-3} H)(10\times 10^{-6} F)}} = 2000 rad/\sec$$

At resonance $Z = 2\Omega$

$$I = \frac{V_S}{Z} = \frac{10\angle 0°}{2} = 5A$$

$$\omega_0 L = (2\times 10^3)(25\times 10^{-3}) = 50\Omega$$

$$V_L = j\omega_0 L I = j50\times 5 = 250\angle 90° (V)$$

At resonance

$$|V_L| = \omega_0 L \left|\frac{V_S}{R}\right| = Q|V_S|$$

$$|V_C| = Q|V_S|$$

LEARNING EXAMPLE

Given L = 0.02H with a Q factor of 200, determine the capacitor necessary to form a circuit resonant at 1000Hz

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow 2\pi \times 1000 = \frac{1}{\sqrt{0.02C}} \Rightarrow C = 1.27\mu F$$

What is the rating for the capacitor if the circuit is tested with a 10V supply?

At resonance

$$|V_L| = \omega_0 L \left|\frac{V_S}{R}\right| = Q|V_S|$$

$$|V_C| = Q|V_S| \quad \Rightarrow |V_C| = 2000V$$

$$\text{L with Q} = 200 \Rightarrow 200 = \frac{\omega_0 L}{R} \Rightarrow R = \frac{2\pi \times 1000 \times 0.02}{200} = 1.59\Omega$$

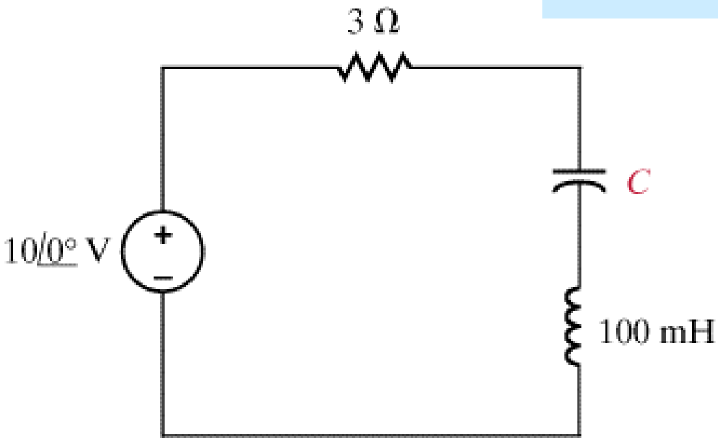
$$I = \frac{10}{1.59} = 6.28A$$

The reactive power on the capacitor exceeds 12kVA

GEAUX

LEARNING EXTENSION

Find the value of C that will place the circuit in resonance at 1800rad/sec



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$1800 = \frac{1}{\sqrt{0.1(H) \times C}} \Rightarrow C = \frac{1}{0.1 \times 1800^2}$$

$$C = 3.86 \mu F$$

Find the Q for the network and the magnitude of the voltage across the capacitor

$$Q = \frac{\omega_0 L}{R}$$

$$Q = \frac{1800 \times 0.1}{3} = 60$$

At resonance

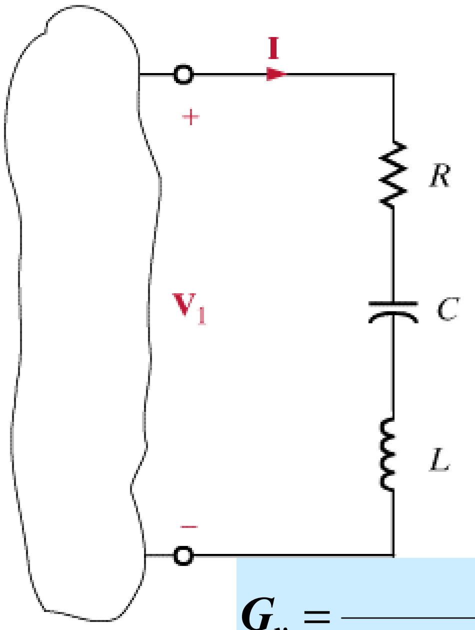
$$|V_L| = \omega_0 L \left|\frac{V_S}{R}\right| = Q|V_S|$$

$$|V_C| = Q|V_S|$$

$$|V_C| = 600V$$

GEAUX

Resonance for the series circuit



$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$

Claim: The voltage gain

$$G_v = \frac{V_R}{V_1} = \frac{1}{1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}}$$

$$G_v = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{Z(j\omega)}$$

At resonance:

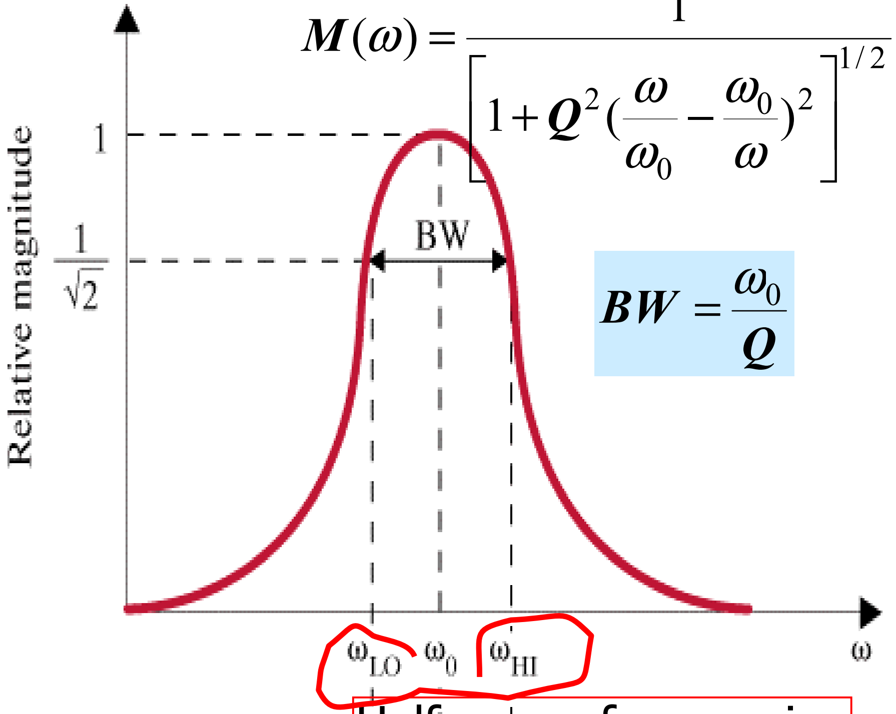
$$\omega_0 L = QR, \ \omega_0 C = \frac{1}{QR}$$

$$Z(j\omega) = R + j\frac{\omega}{\omega_0}QR - j\frac{\omega_0}{\omega}QR$$

$$= R\left[1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})\right]$$

$$G_v = \frac{R}{Z}$$

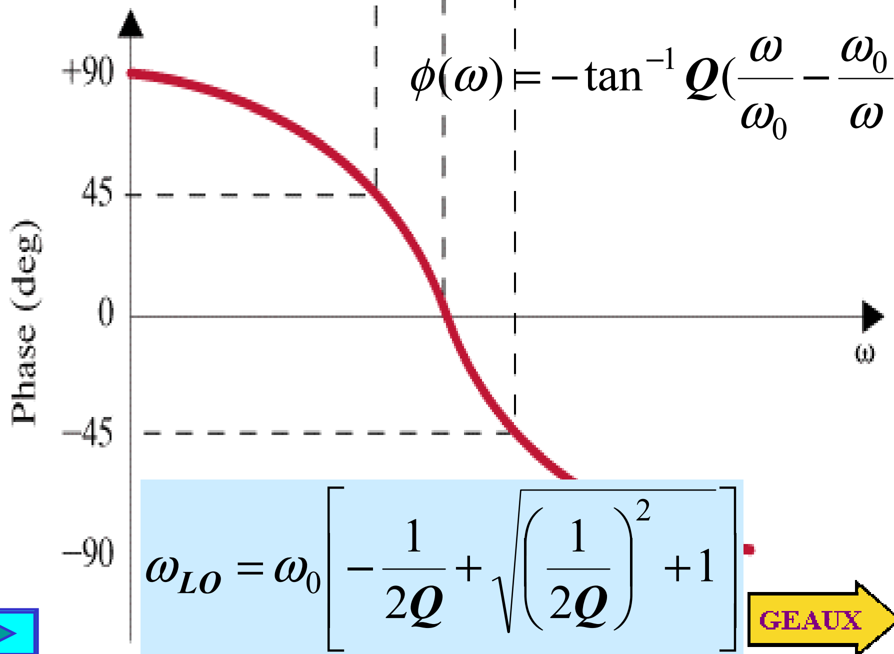
$$M(\omega) = |G_v|, \ \phi(\omega)| = \angle G_v$$



$$M(\omega) = \frac{1}{\left[1 + Q^2(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})^2\right]^{1/2}}$$

$$BW = \frac{\omega_0}{Q}$$

Half power frequencies



$$\phi(\omega) = -\tan^{-1} Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$$

$$\omega_{LO} = \omega_0\left[-\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}\right].$$

GEAUX

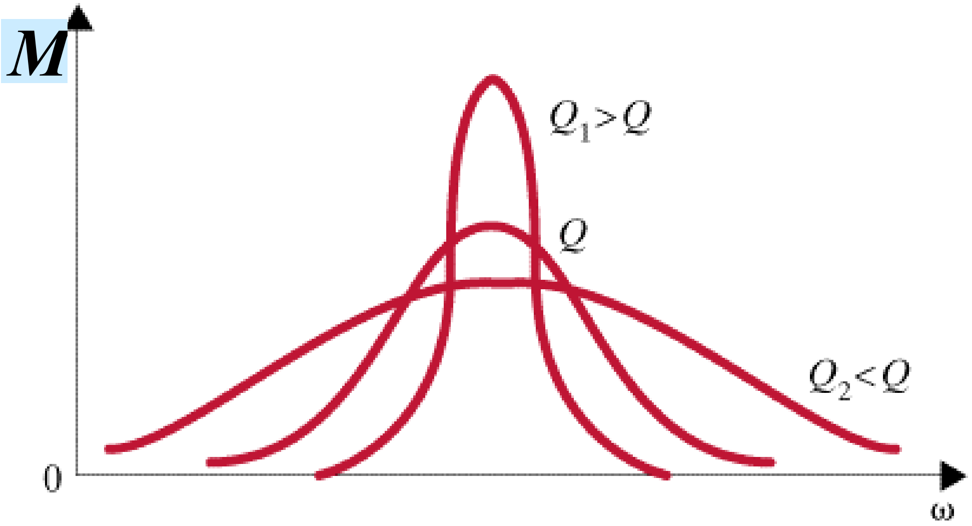
The Q factor

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

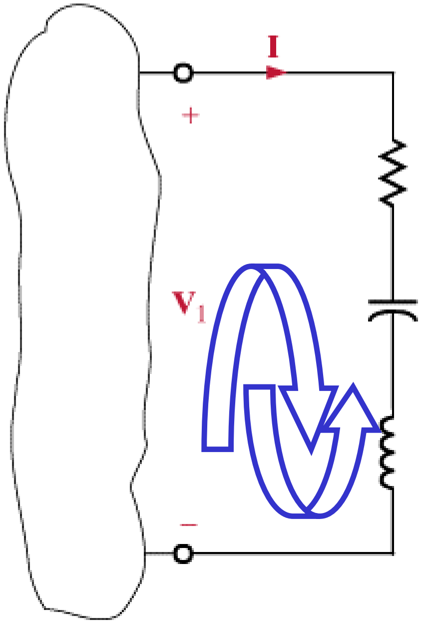
For series circuit : High Q $\Leftrightarrow$ Low R

For parallel circuit : High Q $\Leftrightarrow$ High R (low G)

High Q $\Leftrightarrow$ Small BW



Q can also be interpreted from an energy point of view



dissipates

Stores as E field

Stores as M field

Capacitor and inductor exchange stored energy. When one is at maximum the other is at zero

$$Q = 2\pi \frac{W_S}{W_D} = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated by cycle}}$$

$$W_D = R I_{eff}^2 \times \frac{\omega_0}{2\pi} = \frac{1}{2} R I_{mx}^2 \times \frac{\omega_0}{2\pi}$$

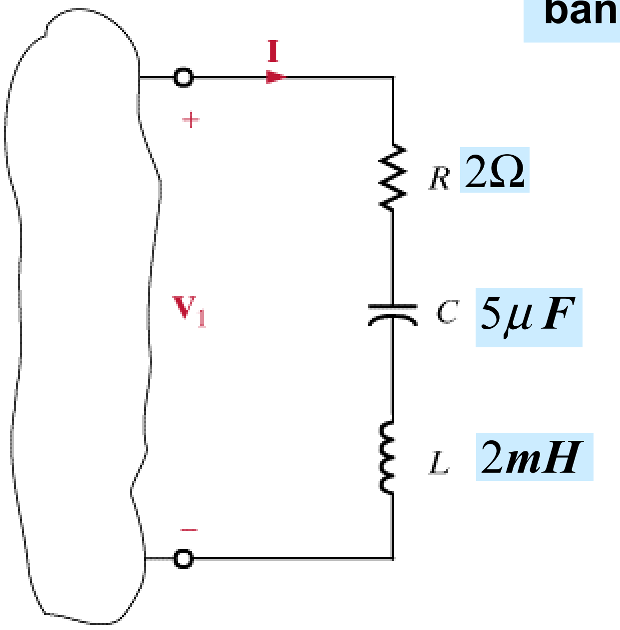
$$W_S = \frac{1}{2} L I_{mx}^2 = \frac{1}{2} C V_{mx}^2$$

$$\frac{W_s}{W_D} = \frac{L\omega_0}{2\pi \times R} = \frac{Q}{2\pi}$$

GEAUX

LEARNING EXAMPLE

Determine the resonant frequency, quality factor and bandwidth when R=2 and when R=0.2



(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} \qquad BW = \frac{\omega_0}{Q}$$

$$\omega_0 = \frac{1}{\sqrt{(2\times10^{-3})(5\times10^{-6})}} = 10^4 \, rad/\sec$$

R	Q
2	10
0.2	100

$$Q = \frac{10000\times0.002}{R}$$

R	Q	BW(rad/sec)
2	10	1000
0.2	100	100

$$BW = 10000/Q$$

Evaluated with EXCEL

GEAUX

LEARNING EXTENSION

A series RLC circuit as the following properties:

$$R = 4\Omega, \omega_0 = 4000 \boldsymbol{rad} / \sec, \boldsymbol{BW} = 100 \boldsymbol{rad} / \sec$$

Determine the values of L,C.

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \qquad BW = \frac{\omega_0}{Q}$$

1. Given resonant frequency and bandwidth determine Q.
2. Given R, resonant frequency and Q determine L, C.

$$Q = \frac{\omega_0}{BW} = \frac{4000}{100} = 40$$

$$L = \frac{QR}{\omega_0} = \frac{40 \times 4}{4000} = 0.040 H$$

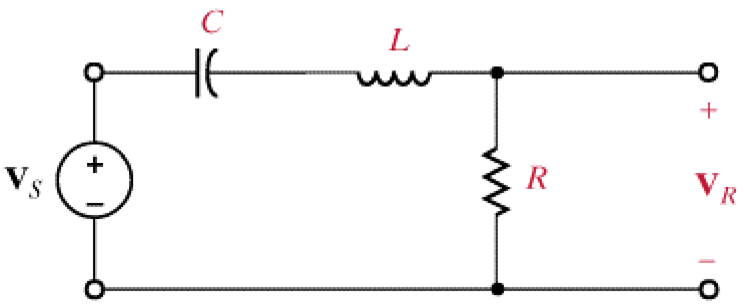
$$C = \frac{1}{L\omega_0^2} = \frac{1}{\omega_0 RQ} = \frac{1}{4 \times 10^{-2} \times 16 \times 10^6} = 1.56 \times 10^{-6} F$$

◀ ▶

GEAUX

LEARNING EXAMPLE

Find R, L, C so that the circuit operates as a band-pass filter with center frequency of 1000rad/s and bandwidth of 100rad/s



$$G_v = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{Z(j\omega)}$$

$$\boxed{\omega_0} = \frac{1}{\sqrt{LC}} \qquad \boxed{Q} = \frac{\boxed{\omega_0} L}{R} = \boxed{\frac{1}{\omega_0 CR}} \qquad \boxed{BW} = \frac{\boxed{\omega_0}}{Q}$$

dependent

Strategy:

1. Determine Q
2. Use value of resonant frequency and Q to set up two equations in the three unknowns
3. Assign a value to one of the unknowns

$$Q = \frac{\omega_0}{BW} = \frac{1000}{100} = 10$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow (10^3)^2 = \frac{1}{LC}$$

$$Q = \frac{\omega_0 L}{R} \Rightarrow 10 = \frac{1000L}{R}$$

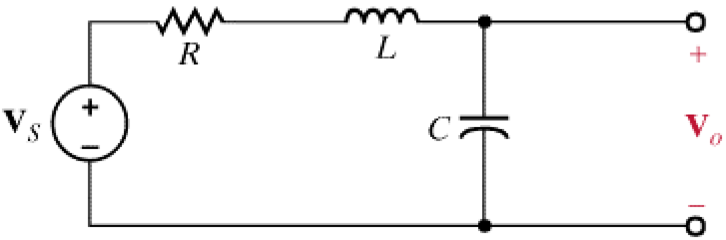
For example $C = 1\mu F = 10^{-6} F$

$L = 1H$

$R = 100\Omega$

GEAUX

PROPERTIES OF RESONANT CIRCUITS: VOLTAGE ACROSS CAPACITOR



At resonance

$$|V_0| = Q\,|V_R|$$

But this is NOT the maximum value for the voltage across the capacitor

$$\left|\frac{V_0}{V_S}\right| = \left|\frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}\right| = \left|\frac{1}{1 - \omega^2 LC + j\omega CR}\right|$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$u = \frac{\omega}{\omega_0}; g = \left|\frac{V_0}{V_S}\right|^2$$

$$g(u) = \frac{1}{\left[\left(1-u^2\right)^2 + \left(\frac{u}{Q}\right)^2\right]}$$

$$\frac{dg}{du} = 0 = \frac{2(1-u^2)(-2u) + 2(u/Q)(1/Q)}{\left[\left(1-u^2\right)^2 + \left(\frac{u}{Q}\right)^2\right]^2}$$

$$\Rightarrow 2(1-u^2) = \frac{1}{Q^2}$$

$$u_{\max} = \frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$$

$$g_{\max} = \frac{1}{\frac{1}{4Q^4} + \left(\frac{1}{Q^2} - \frac{1}{2Q^4}\right)} = \frac{Q^2}{1 - \frac{1}{4Q^2}}$$

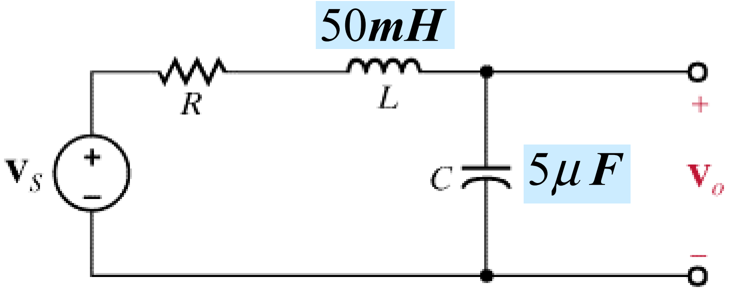
$$|V_0| = \frac{Q\,|V_S|}{\sqrt{1 - \frac{1}{4Q^2}}}$$

◀ ▶

GEAUX

LEARNING EXAMPLE

Determine ω_0, $\omega_{\max}$ when $R = 50\Omega$ and $R = 1\Omega$



Natural frequency depends only on L, C.
Resonant frequency depends on Q.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$u_{\max} = \frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5\times10^{-2})(5\times10^{-6})}} = 2000 rad / s$$

$$Q = \frac{2000\times0.050}{R}$$

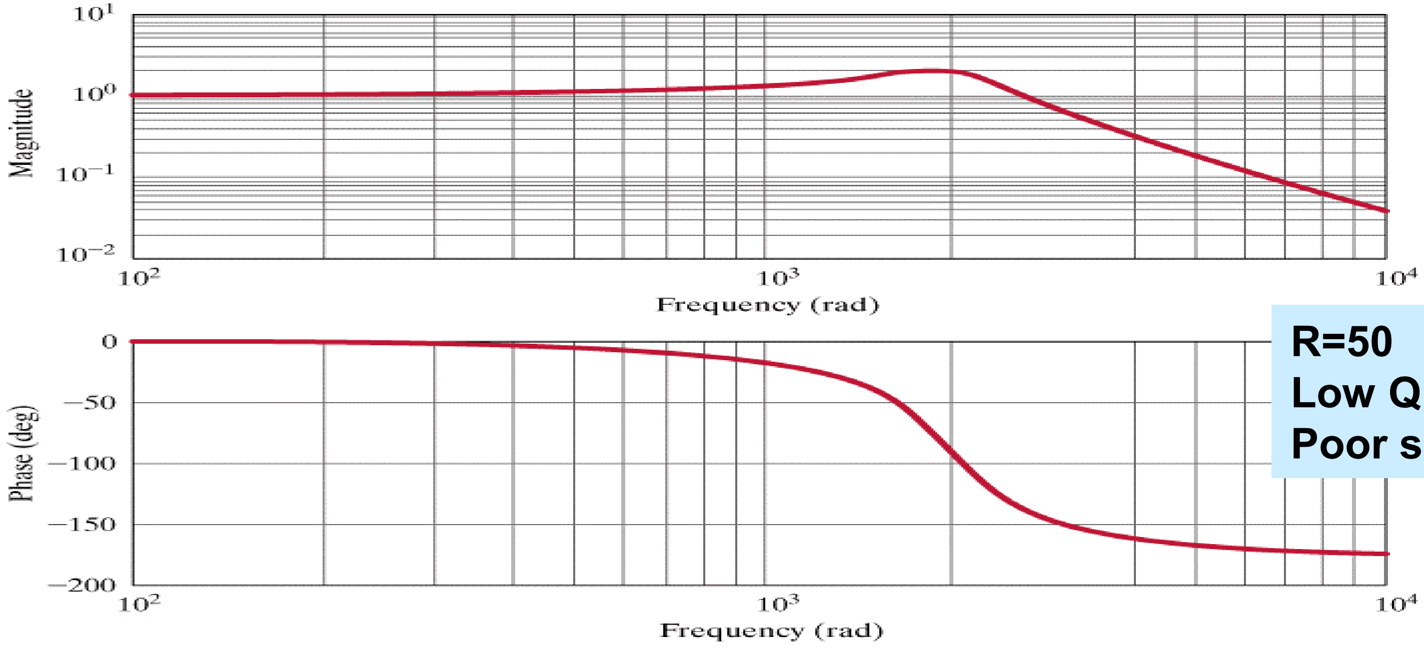
$$\omega_{\max} = 2000\times\sqrt{1 - {}^{1}\!/_{2Q^2}}$$

R	Q	Wmax
50	2	1871
1	100	2000

Evaluated with EXCEL and rounded to zero decimals

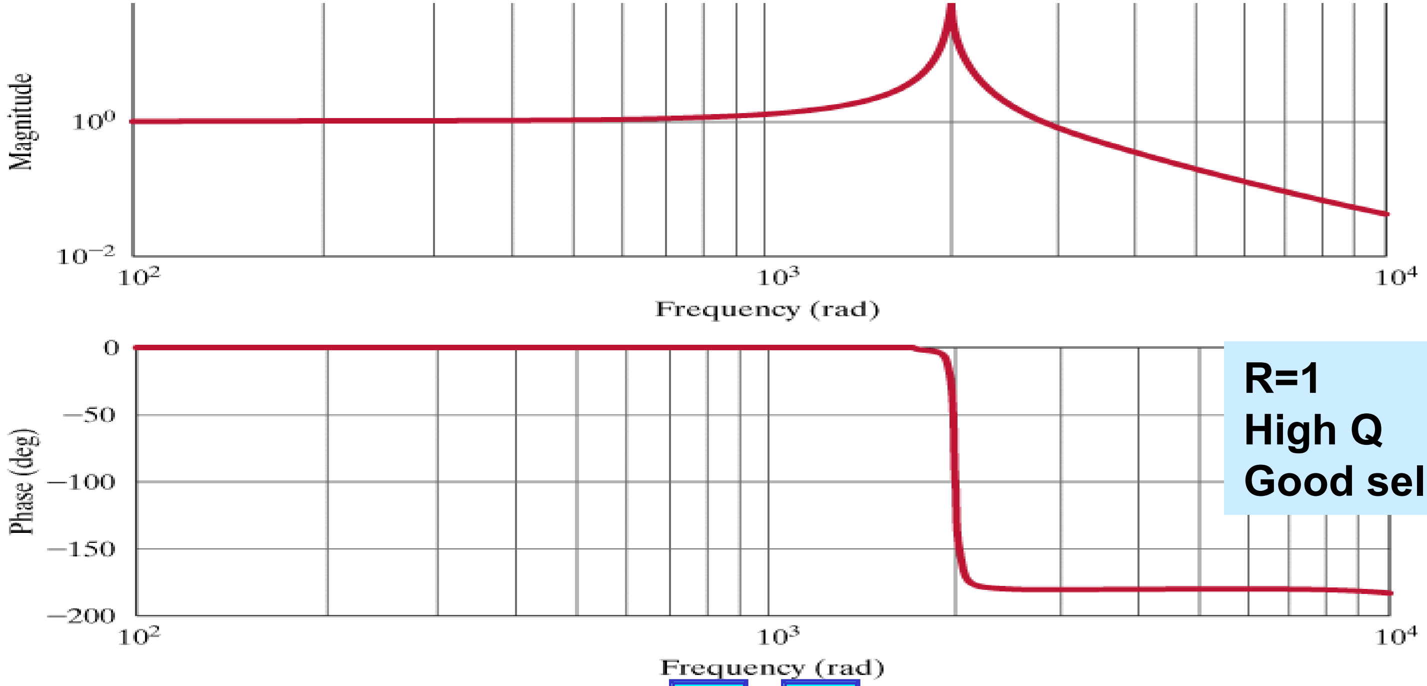
Using MATLAB one can display the frequency response

GEAUX



R=50
Low Q
Poor selectivity

(a)



R=1
High Q
Good selectivity

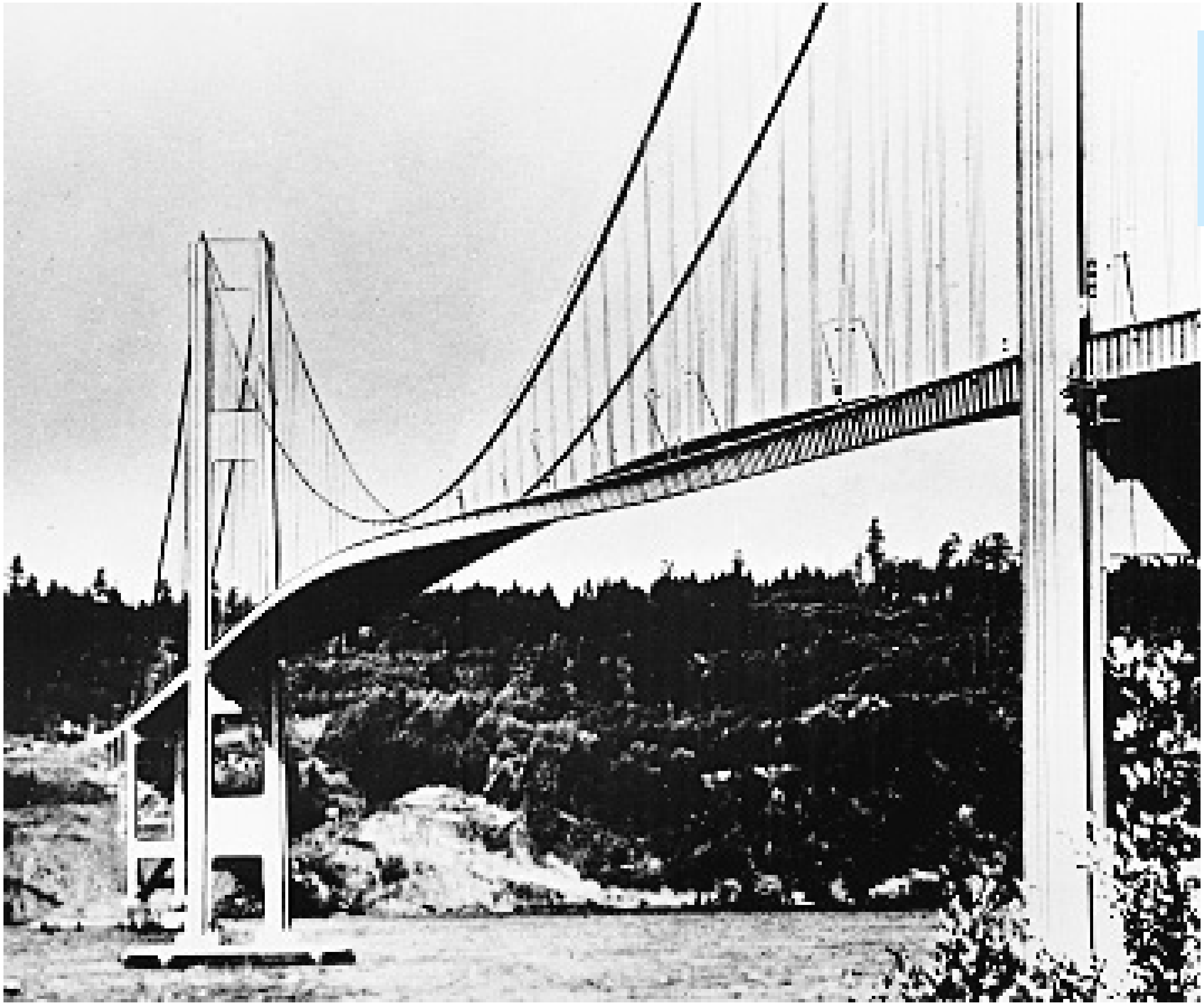
(b

GEAUX

LEARNING EXAMPLE

The Tacoma Narrows Bridge

Opened: July 1, 1940
Collapsed: Nov 7, 1940

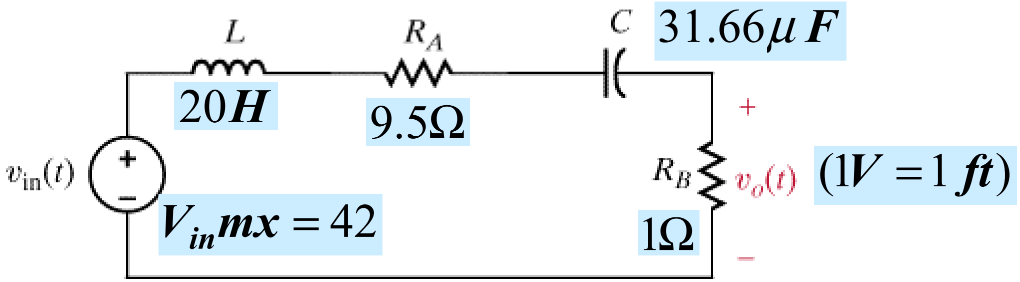


Likely cause: wind varying at frequency similar to bridge natural frequency

$$\omega_0 = 2\pi \times 0.2$$

Tacoma Narrows Bridge Simulator

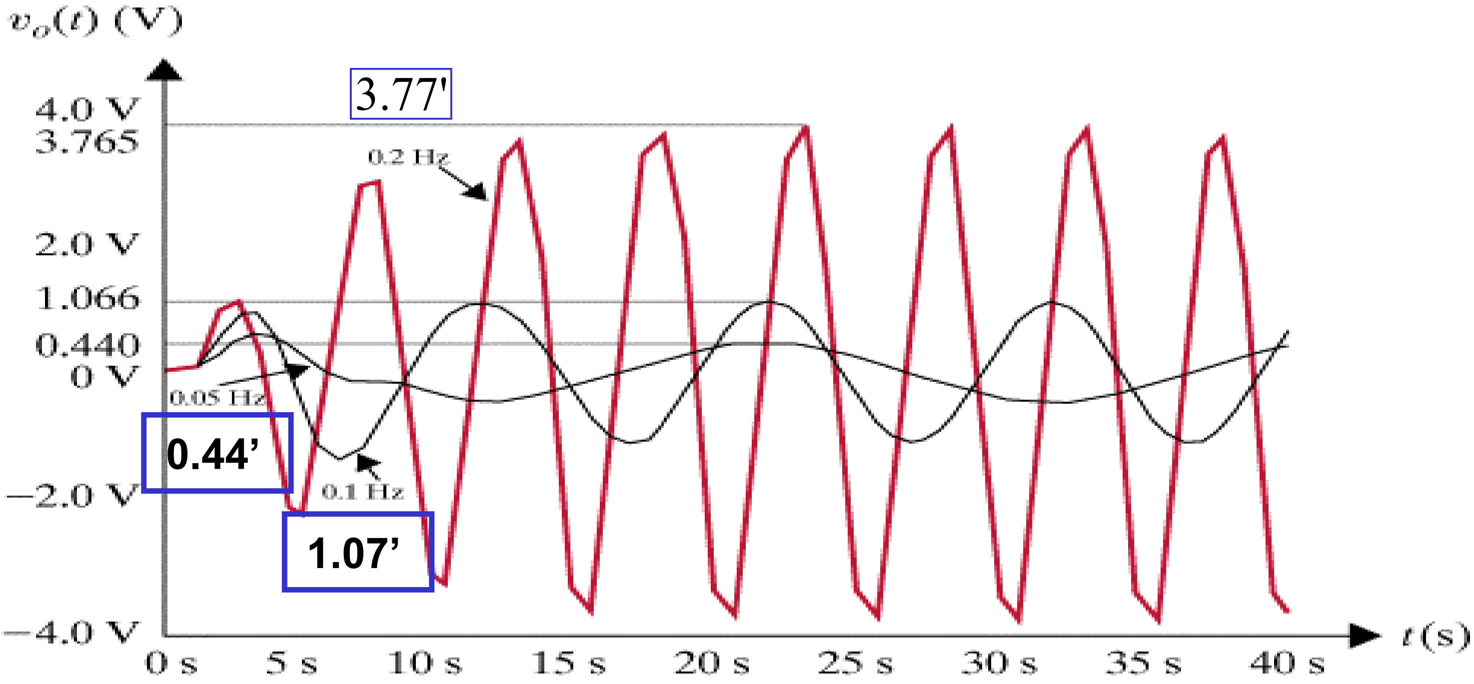
Assume a low Q=2.39



At failure a 42mph wind caused 4' deflection.

For the model at resonance

$$\frac{v_0}{v_{in}} = \frac{R_B}{R_A + R_B} \approx \frac{4}{42}$$



GEAUX

PARALLEL RLC RESONANT CIRCUITS

Impedance of series RLC

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$

Notice equivalences

$$R \leftrightarrow G, L \leftrightarrow C, C \leftrightarrow L$$

$$Z \leftrightarrow Y, V \leftrightarrow I$$

Admittance of parallel RLC

$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

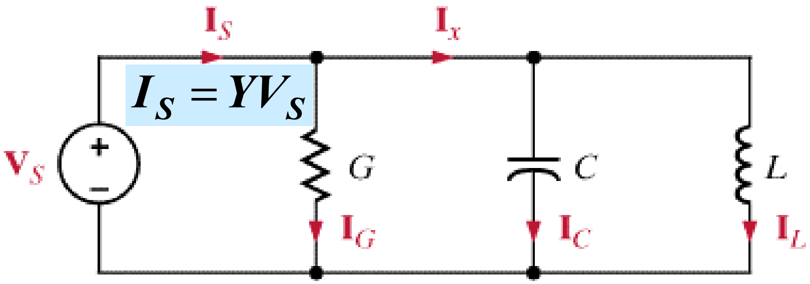
$$|Y|^2 = G^2 + (\omega C - \frac{1}{\omega L})^2$$

Series RLC

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Parallel RLC

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG}$$



$$I_S = YV_S$$

$$I_G = GV_S = \frac{G}{Y} I_S$$

$$I_C = j\omega C V_S = \frac{j\omega C}{Y} I_S$$

$$I_L = \frac{1}{j\omega L} V_S = \frac{1/j\omega L}{Y} I_S$$

At resonance

$$\omega_0 C = \frac{1}{\omega_0 L} \Rightarrow Y = G$$

$$I_G = I_S$$

$$I_C = -I_L$$

$$|I_C| = \frac{\omega_0 C}{G} |I_S| = Q|I_S|$$

$$|I_L| = \frac{1}{\omega_0 LG} |I_S|$$

Series RLC $BW = \frac{\omega_0}{Q}$

Parallel RLC $BW = \frac{\omega_0}{Q}$

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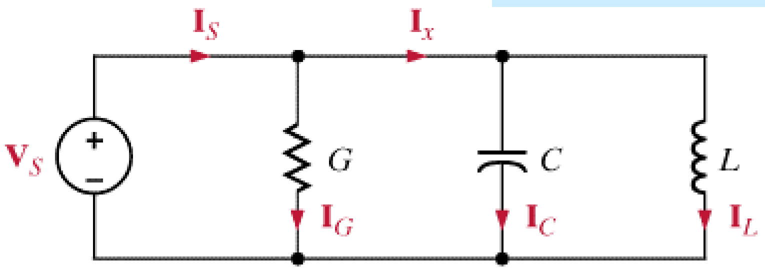
LEARNING EXAMPLE

If the source operates at the resonant frequency of the network, compute all the branch currents



$$V_S = 120\angle 0^\circ, \quad G = 0.01S$$
$$C = 600\mu F, \quad L = 120mH$$

$$I_G = 0.01 \times 120\angle 0^\circ = 1.2\angle 0^\circ (A) = I_S$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.120 \times (6\times 10^{-4})}} = 117.85 rad/s$$

$$I_C = (1\angle 90^\circ)\times(117.85)\times(600\times 10^{-6})\times 120\angle 0^\circ = 8.49\angle 90^\circ (A)$$

$$I_L = 8.49\angle -90^\circ (A)$$

$I_x =$ ________

$$\boxed{I_G = GV_S = \frac{G}{Y} I_S}$$

$$\boxed{I_C = j\omega C V_S} = \frac{j\omega C}{Y} I_S$$

$$I_L = \frac{1}{j\omega L} V_S = \frac{1/j\omega L}{Y} I_S$$

At resonance

$$\boxed{\omega_0 C = \frac{1}{\omega_0 L}} \Rightarrow Y = G$$

$$I_G = I_S$$

$$\boxed{I_C = -I_L}$$

$$|I_C| = \frac{\omega_0 C}{G} |I_S| = Q |I_S|$$

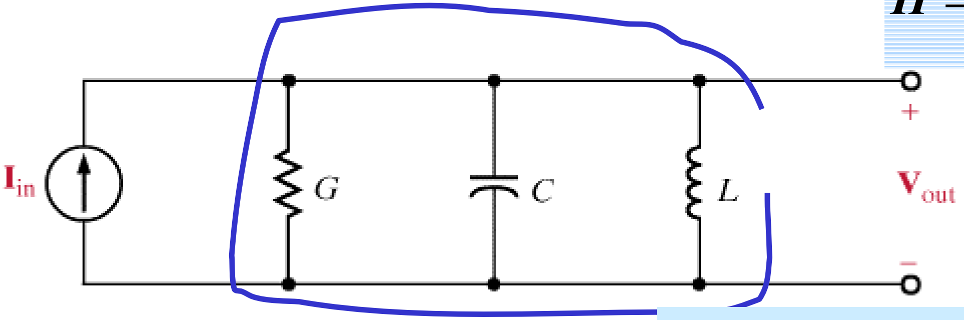
$$|I_L| = \frac{1}{\omega_0 L G} |I_S|$$

GEAUX

LEARNING EXAMPLE

Derive expressions for the resonant frequency, half power frequencies, bandwidth and quality factor for the transfer characteristic

$$H = \frac{V_{out}}{I_{in}}$$



$$V_{out} = \frac{I_{in}}{Y_T} \Rightarrow H = \frac{V_{out}}{I_{in}} = \frac{1}{Y_T}$$

$$Y_T = G + j\omega C + \frac{1}{j\omega L}$$

$$|H| = \left| \frac{1}{G + j\omega C + \frac{1}{j\omega L}} \right| = \frac{1}{\sqrt{G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}}$$

Resonant frequency : $\omega_0 = \frac{1}{\sqrt{LC}}$

$$|H_{\max}| = \frac{1}{G} = R$$

Half power frequencies $\Rightarrow |H(j\omega_h)|^2 = 0.5\,|H|^2_{\max}$

$$G^2 + \left(\omega_h C - \frac{1}{\omega_h L}\right)^2 = 2G^2$$

$$\Rightarrow \omega_h C - \frac{1}{\omega_h L} = \pm G$$

$$\omega_h = \mp \frac{G}{2C} + \sqrt{\left(\frac{G}{2C}\right)^2 + \frac{1}{LC}}$$

$$BW = \omega_{HI} - \omega_{LO} = \frac{G}{C}$$

$$Q = \frac{\omega_0}{BW} = \frac{1}{G}\sqrt{\frac{C}{L}} = R\sqrt{\frac{C}{L}}$$

Replace and show

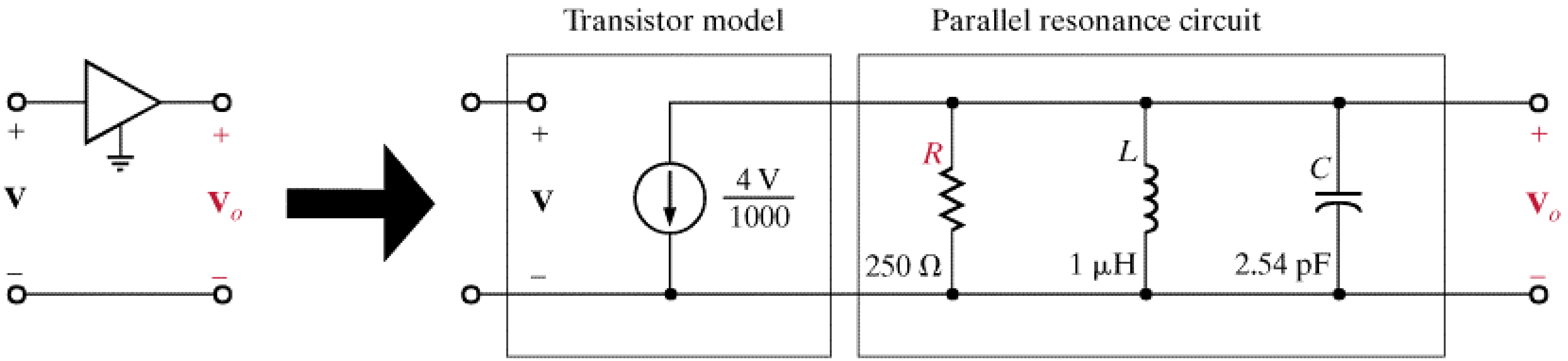
$$Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 L G}$$

$$\omega_{LO} = \omega_0 \left[-\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \right]$$

GEAUX

LEARNING EXAMPLE

Increasing selectivity by cascading low Q circuits

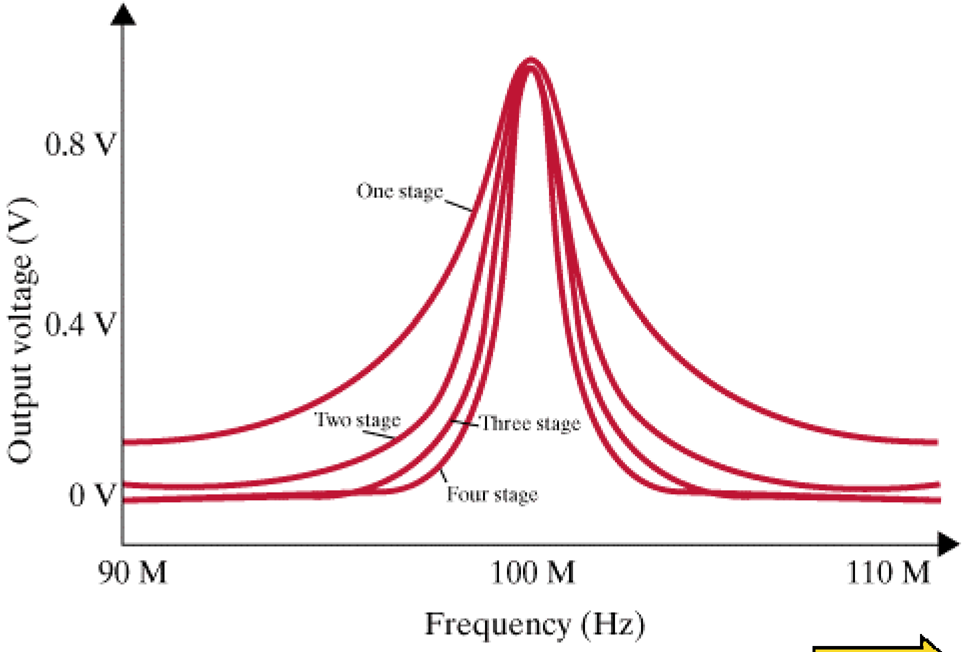


Single stage tuned amplifier

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10^{-6}\,H)(2.54\times10^{-12}\,F)}} = 6.275\times10^{8}\,rad/s = 99.9\,MHz$$

$$Q = \frac{\omega_0}{BW} = \frac{1}{G}\sqrt{\frac{C}{L}} = R\sqrt{\frac{C}{L}}$$

$$= 250\times\sqrt{\frac{2.54\times10^{-12}}{10^{-6}}} = 0.398$$



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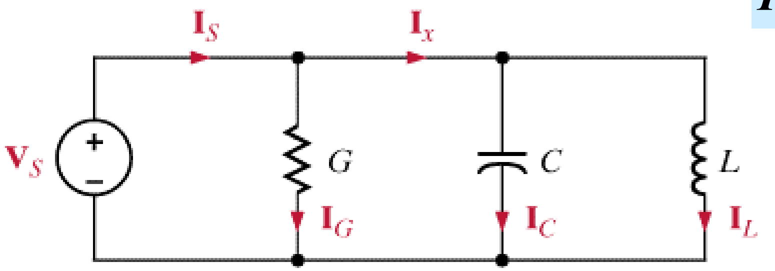
LEARNING EXTENSION

Determine the resonant frequency, Q factor and bandwidth

$$R = 2k\Omega, L = 20mH, C = 150\mu F$$



Parallel RLC

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG} \qquad BW = \frac{\omega_0}{Q}$$

$$\omega_0 = \frac{1}{\sqrt{(20\times10^{-3})(150\times10^{-6})}} = 577 rad/s$$

$$Q = \frac{577\times150\times10^{-6}}{(1/2000)} = 173$$

$$BW = \frac{577}{173} = 3.33 rad/s$$

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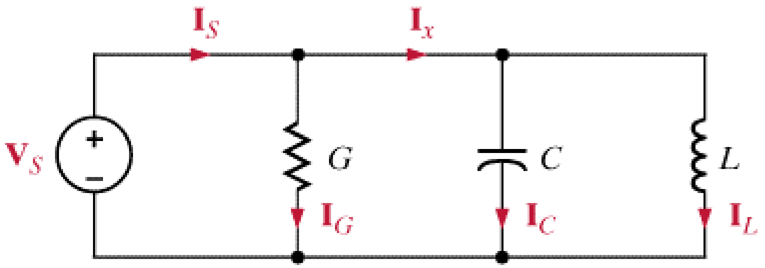
LEARNING EXTENSION

Determine L, C, ω_0



$R = 6k\Omega, BW = 1000 rad/s, Q = 120$

Parallel RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG}$$

$$BW = \frac{\omega_0}{Q}$$

$$\omega_0 = Q \times BW = 120 \times 1000 = 1.2 \times 10^5 rad/s$$

$$C = \frac{Q}{R\omega_0} = \frac{120}{6000 \times 1.2 \times 10^5} = 0.167 \mu F$$

$$L = \frac{R}{Q\omega_0} = \frac{6000}{120 \times 1.2 \times 10^5} = 417 \mu H$$

Can be used to verify computations

◀ ▶

GEAUX

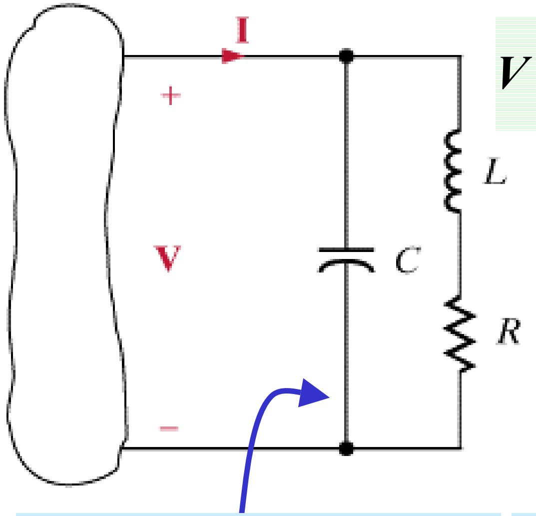
PRACTICAL RESONANT CIRCUIT

The resistance of the inductor coils cannot be neglected

$V = ZI = \frac{I}{Y}$. At resonance the voltage and impedance are maxima

$$Z_{MAX} = \frac{R^2 + (\omega_R L)^2}{R} = R\left(1 + \left(\frac{\omega_R L}{R}\right)^2\right) = R\left(1 + \left(\frac{\omega_R}{\omega_0}\right)^2 \left(\frac{\omega_0 L}{R}\right)^2\right)$$

$$Z_{MAX} = RQ_0^2$$



$$Y(j\omega) = j\omega C + \frac{1}{R + j\omega L} \times \frac{R - j\omega L}{R - j\omega L}$$

$$Y(j\omega) = j\omega C + \frac{R - j\omega L}{R^2 + (\omega L)^2}$$

$$Y(j\omega) = \frac{R}{R^2 + (\omega L)^2} + j\left(C - \frac{\omega L}{R^2 + (\omega L)^2}\right)$$

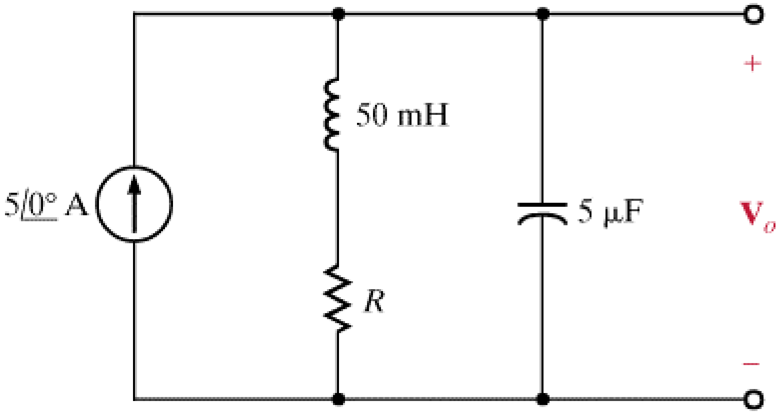
$$Y \text{ real} \Rightarrow C - \frac{\omega L}{R^2 + (\omega L)^2} = 0 \Rightarrow \omega_R = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}},\ Q_0 = \frac{\omega_0 L}{R} \Rightarrow \omega_R = \omega_0 \sqrt{1 - \frac{1}{Q_0^2}}$$

How do you define a quality factor for this circuit?

LEARNING EXAMPLE

Determine both ω_0, ω_R for $R = 50\Omega, 5\Omega$

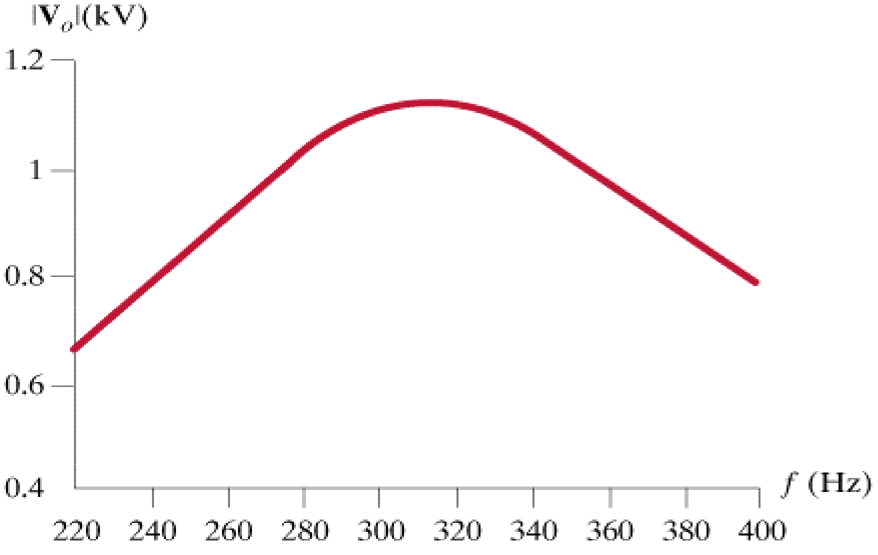


$$\omega_0 = \frac{1}{\sqrt{LC}}, \ Q_0 = \frac{\omega_0 L}{R} \Rightarrow \omega_R = \omega_0 \sqrt{1 - \frac{1}{Q_0^2}}$$

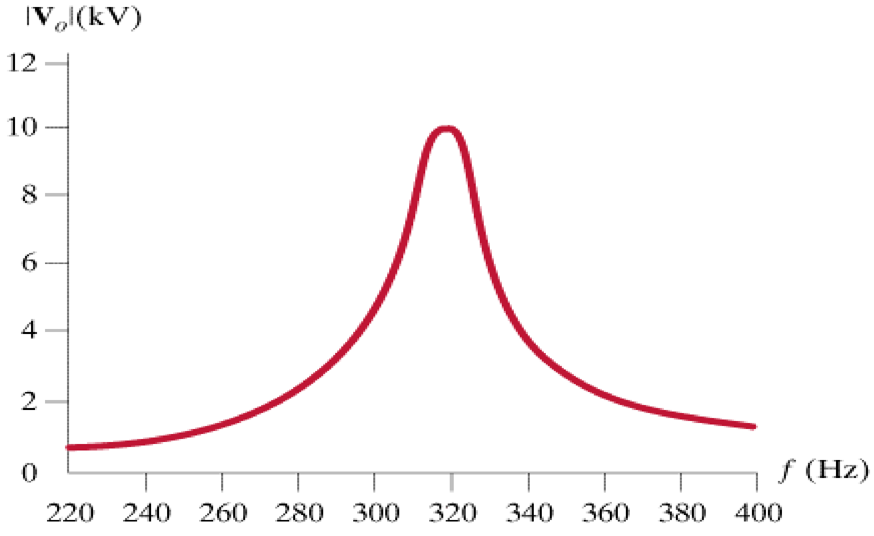
$$\omega_0 = \frac{1}{\sqrt{(50\times10^{-3}H)(5\times10^{-6}F)}} = 2000 rad/s$$

$$Q_0 = \frac{2000\times0.050}{R}, \ \omega_R = 2000\sqrt{1 - \frac{1}{Q_0^2}}$$

R	Q0	Wr(rad/s)	f(Hz)
50	2	1732	275.7
5	20	1997	317.8



(a) $R = 50\ \Omega$



(b) $R = 5\ \Omega$

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RESONANCE IN A MORE GENERAL VIEW

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$

$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

$$|Y|^2 = G^2 + (\omega C - \frac{1}{\omega L})^2$$

For series connection the impedance reaches maximum at resonance. For parallel connection the impedance reaches maximum

$$Y_s = \frac{j\omega C}{(j\omega)^2 LC + j\omega CR + 1} \qquad Z_p = \frac{j\omega L}{(j\omega)^2 LC + j\omega LG + 1}$$

In Bode plots the quadratic term was written as

$$(j\omega\tau)^2 + 2\varsigma\, j\omega\tau + 1$$

$$\tau = \sqrt{LC} = \frac{1}{\omega_0}$$

series

$$2\varsigma\tau = CR \Rightarrow 2\varsigma = \omega_0 CR = \frac{1}{Q}$$

parallel

$$2\varsigma\tau = LG \Rightarrow 2\varsigma = \omega_0 LG = \frac{1}{Q}$$

A high Q circuit is highly under damped

$$Q = \frac{1}{2\varsigma}$$

Resonance
STOP
GEAUX
◀ ▶

SCALING

Scaling techniques are used to change an idealized network into a more realistic one or to adjust the values of the components

Magnitude or impedance scaling

$$R' \rightarrow K_M R$$

$$L' \rightarrow K_M L$$

$$C' \rightarrow \frac{C}{K_M}$$

$$LC = L'C' \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{L'C'}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L'}{R'}$$

Magnitude scaling does not change the frequency characteristics nor the quality of the network.

Frequency or time scaling

$$\omega' \rightarrow K_F \omega$$

Impedance of each component is unchanged

$$\omega' L' = \omega L, \quad \frac{1}{\omega' C'} = \frac{1}{\omega C}$$

$$R' \rightarrow R$$

$$L' \rightarrow \frac{L}{K_F}$$

$$C' \rightarrow \frac{C}{K_F}$$

$$\omega_0' = K_F \omega_0$$

$$Q' = \frac{\omega_0' L'}{R'} = Q$$

Constant Q networks

$$BW' = \frac{\omega_0'}{Q'} = K_F (BW)$$

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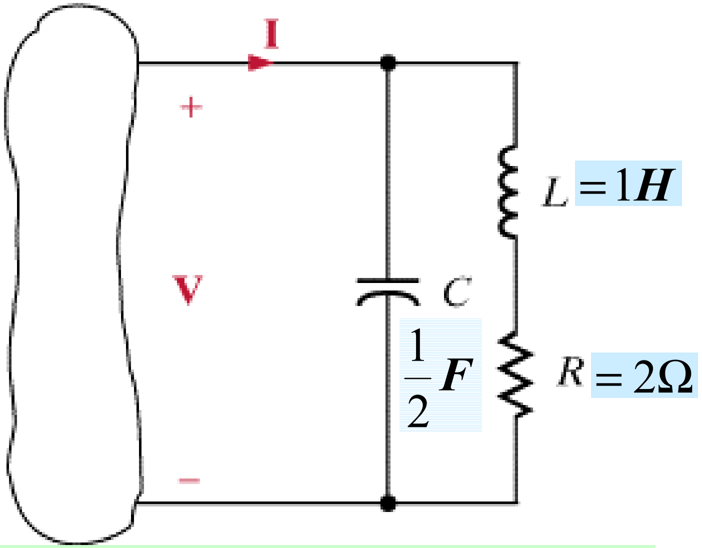
LEARNING EXAMPLE

Determine the value of the elements and the characterisitcs of the network if the circuit is magnitude scaled by 100 and frequency scaled by 1,000,000



$$\omega_0 = \sqrt{2}\, rad/s, \quad Q = \frac{\sqrt{2}}{2}, \quad BW = 2$$

Magnitude or impedance scaling

$$R' \to K_M R$$
$$L' \to K_M L$$
$$C' \to \frac{C}{K_M}$$

$$R' = 200\Omega$$
$$L' = 100H$$
$$C' = \frac{1}{200}F$$

Q, ω_0 are unchanged

$$R' \to R$$
$$L' \to \frac{L}{K_F}$$
$$C' \to \frac{C}{K_F}$$

$$R'' = 200\Omega$$
$$L'' = 100mH$$
$$C'' = \frac{1}{200}\mu F$$

$$\omega_0' = K_F \omega_0 \qquad \omega_0'' = 1.414 \times 10^6\, rad/s$$

$$BW' = \frac{\omega_0'}{Q'} = K_F (BW)$$

GEAUX

LEARNING EXTENSION

An RLC network with $R = 10\Omega$, $L = 1H$, $C = 2F$ is magnitude scaled by 100 and frequency scaled by 10,000. Determine the resulting circuit elements

Magnitude or impedance scaling

$$R' \rightarrow K_M R$$

$$L' \rightarrow K_M L$$

$$C' \rightarrow \frac{C}{K_M}$$

$$R' = 1000\Omega$$

$$L' = 100H$$

$$C' = 0.02F$$

Frequency scaling

$$R' \rightarrow R$$

$$L' \rightarrow \frac{L}{K_F}$$

$$C^{'} \rightarrow \frac{C}{K_F}$$

$$R'' = 1k\Omega$$

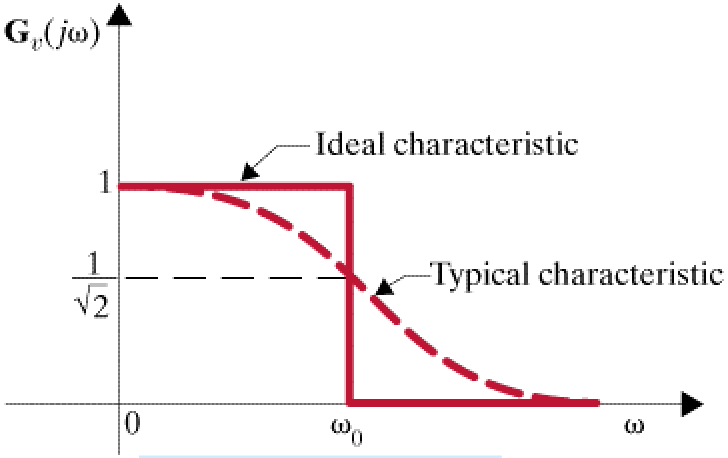
$$L'' = 0.01H$$

$$C'' = 2\mu F$$

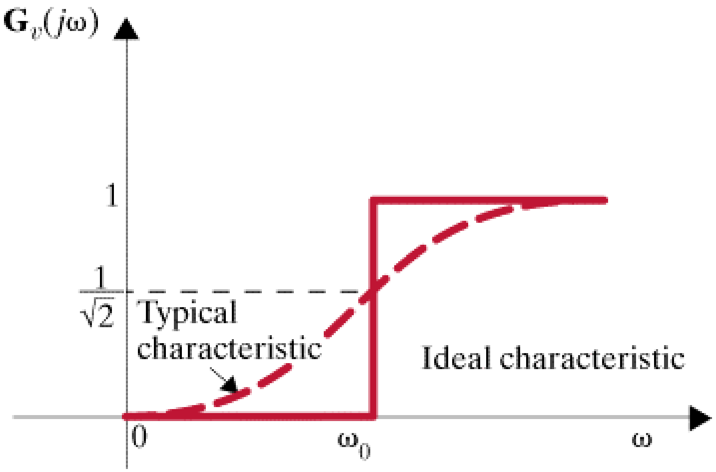
FILTER NETWORKS

Networks designed to have frequency selective behavior

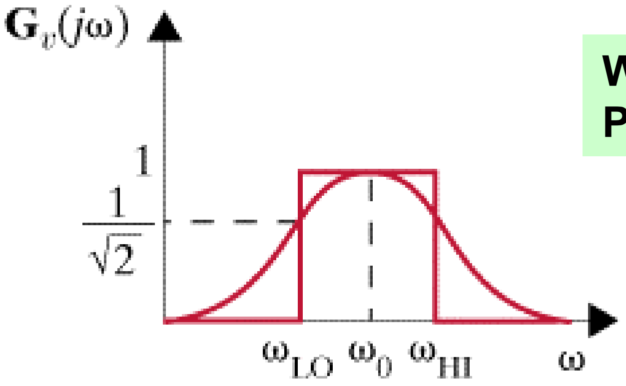
COMMON FILTERS



Low-pass filter

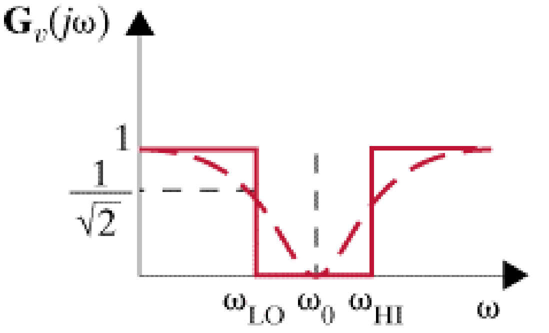


High-pass filter



Band-pass filter

We focus first on PASSIVE filters



Band-reject filter

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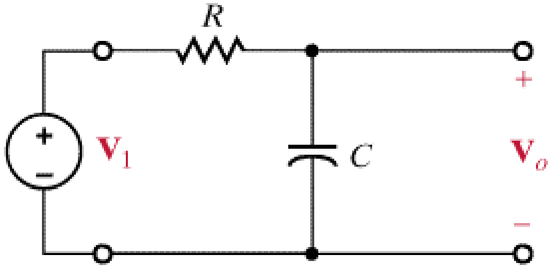
Simple low-pass filter



(b)

$$G_v = \frac{V_0}{V_1} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

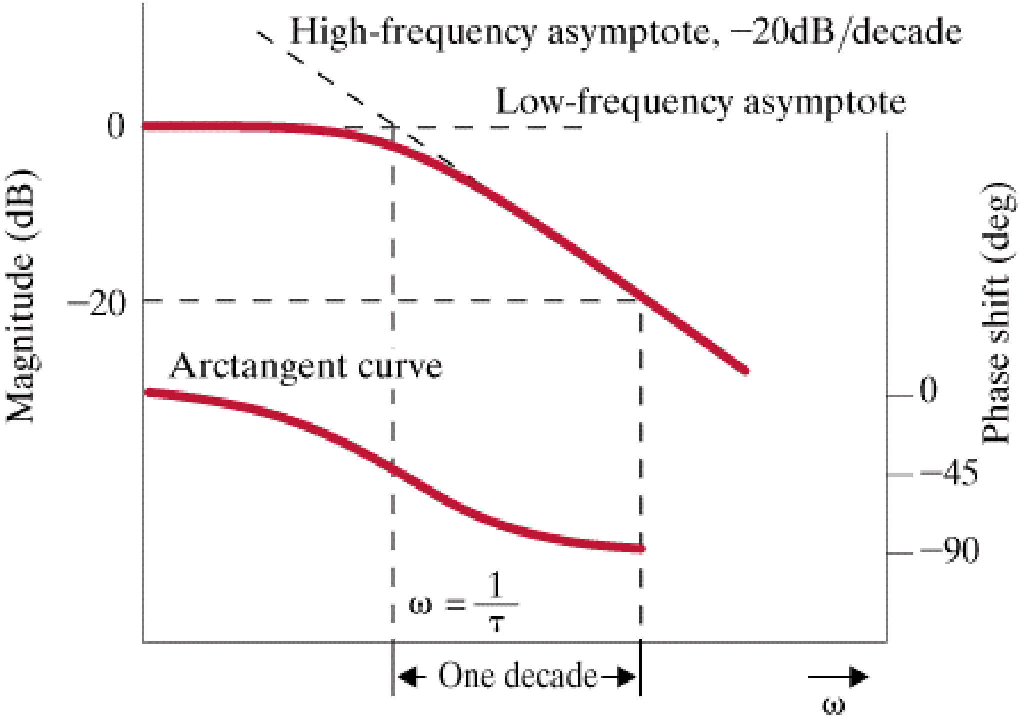
$$G_v = \frac{1}{1 + j\omega\tau}; \quad \tau = RC$$

$$M(\omega) = |G_v| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = -\tan^{-1}\omega\tau$$

$$M_{\max} = 1, \; M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

$\omega = \dfrac{1}{\tau}$ =half power frequency

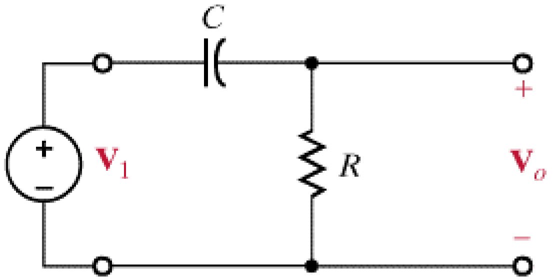


(c)

$$BW = \frac{1}{\tau}$$

GEAUX

Simple high-pass filter



$$G_v = \frac{V_0}{V_1} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega CR}{1 + j\omega CR}$$

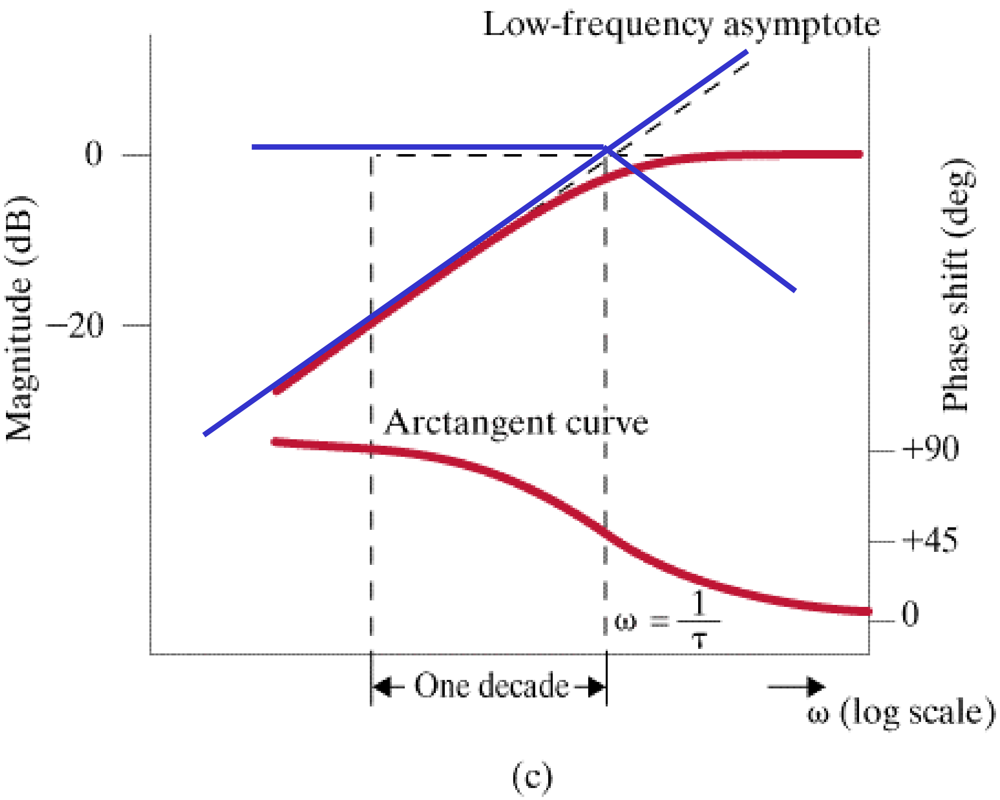
$$G_v = \frac{j\omega\tau}{1 + j\omega\tau}; \ \tau = RC$$

$$M(\omega) = |G_v| = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = \frac{\pi}{2} - \tan^{-1}\omega\tau$$

$$M_{\max} = 1, \ M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

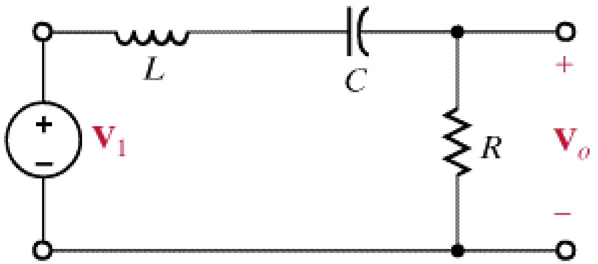
$\omega = \frac{1}{\tau}$ =half power frequency

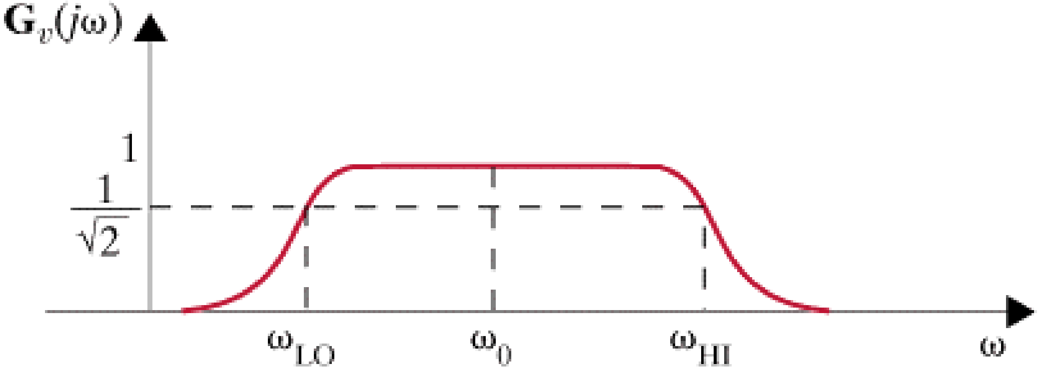


(c)

$$\omega_{LO} = \frac{1}{\tau}$$

Simple band-pass filter





(e)

Band-pass

$$G_v = \frac{V_0}{V_1} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$M(\omega) = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}}$$

$$M\left(\omega = \frac{1}{\sqrt{LC}}\right) = 1 \quad M(\omega = 0) = M(\omega = \infty) = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$M(\omega_{LO}) = \frac{1}{\sqrt{2}} = M(\omega_{HI})$$

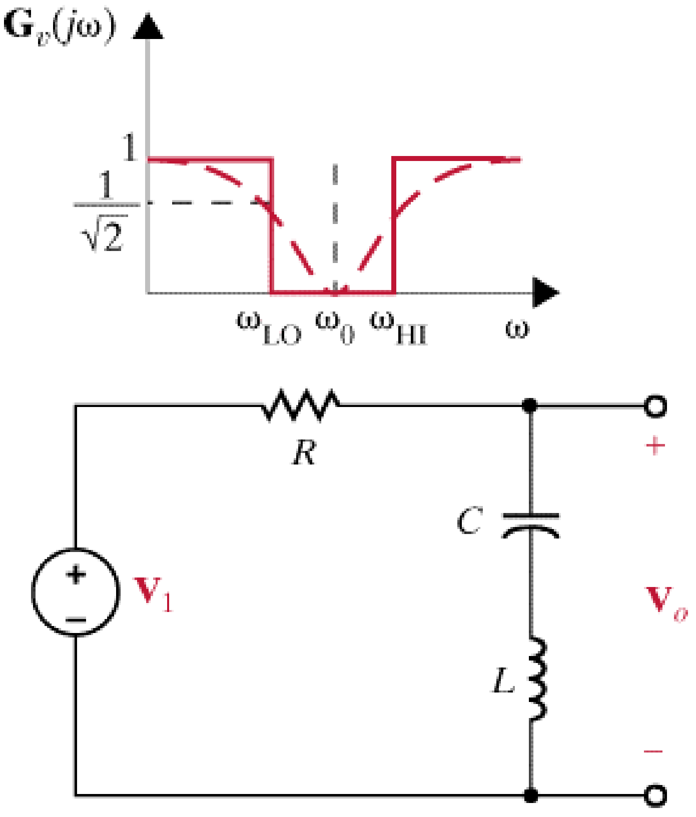
$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

$$\omega_{HI} = \frac{(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

$$BW = \omega_{HI} - \omega_{LO} = \frac{R}{L}$$

GEAUX

Simple band-reject filter



(d)

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow j\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0$$

at $\omega = 0$ the capacitor acts as open circuit $\Rightarrow V_0 = V_1$

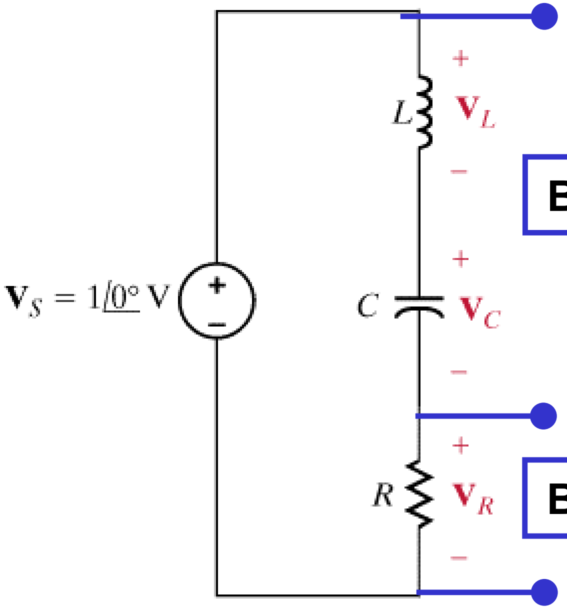
at $\omega = \infty$ the inductor acts as open circuit $\Rightarrow V_0 = V_1$

ω_{LO}, ω_{HI} are determined as in the band-pass filter

GEAUX

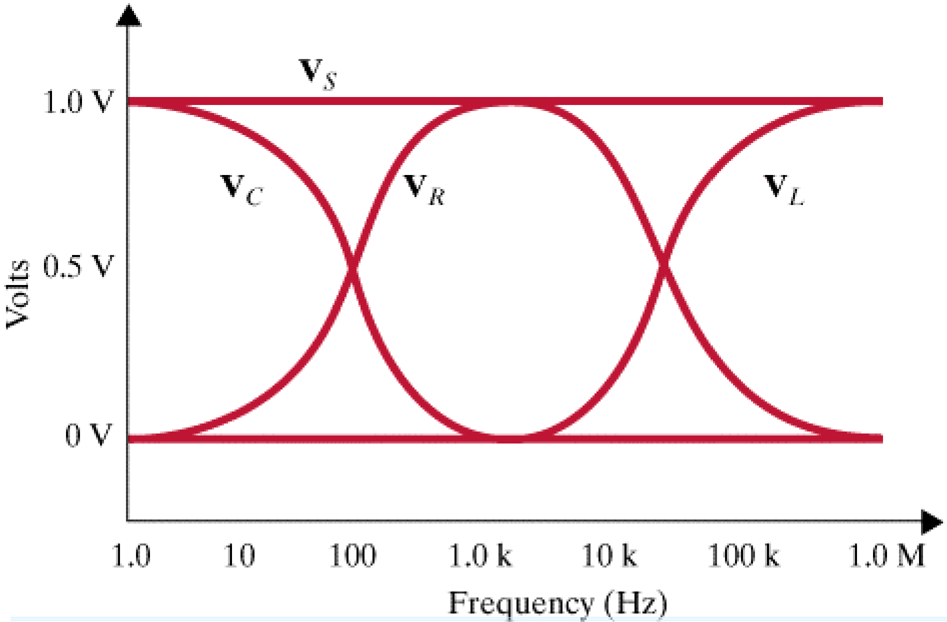
LEARNING EXAMPLE

Depending on where the output is taken, this circuit can produce low-pass, high-pass or band-pass or band-reject filters



Band-reject filter

Band-pass



Bode plot for $R = 10\Omega,\ L = 159\mu H,\ C = 159\mu F$

$$\frac{V_L}{V_S} = \frac{j\omega L}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\frac{V_L}{V_S}(\omega = 0) = 0,\ \frac{V_L}{V_S}(\omega = \infty) = 1$$

High-pass

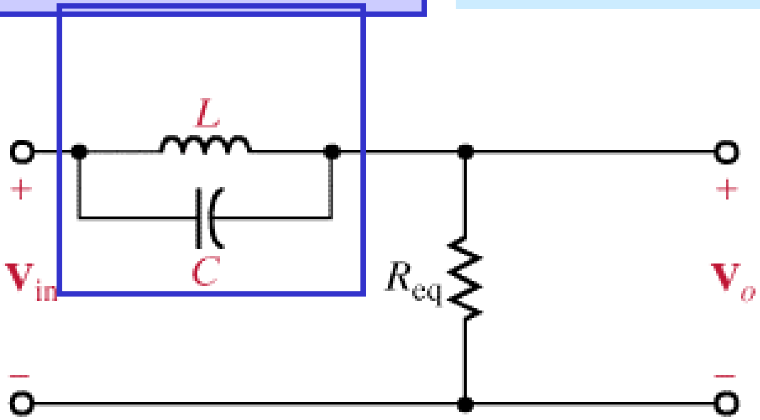
$$\frac{V_C}{V_S} = \frac{\frac{1}{j\omega C}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\frac{V_C}{V_S}(\omega = 0) = 1,\ \frac{V_C}{V_S}(\omega = \infty) = 0$$

Low-pass

LEARNING EXAMPLE

A simple notch filter to eliminate 60Hz interference



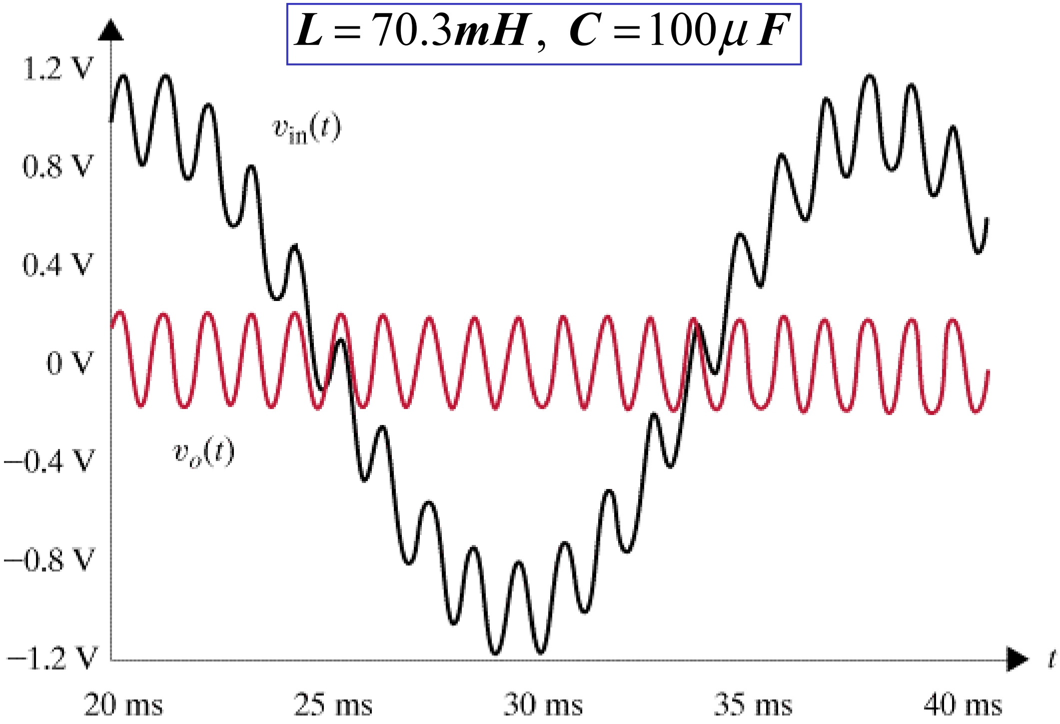
$$Z_R=\frac{j\omega L\frac{1}{j\omega C}}{j\omega L+\frac{1}{j\omega C}}=\frac{L/C}{j(\omega L-\frac{1}{\omega C})}$$

$$V_0=\frac{R_{eq}}{R_{eq}+Z_R}V_{in}$$

$$Z_R\left(\omega=\frac{1}{\sqrt{LC}}\right)=\infty \qquad \therefore V_0\left(\omega=\frac{1}{\sqrt{LC}}\right)=0$$

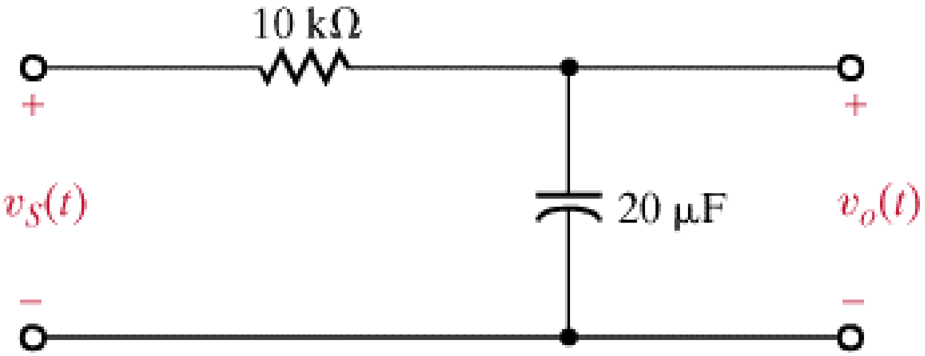
$$v_{in}(t)=\sin(2\pi\times 60t)+0.2\sin(2\pi\times 1000t)$$

$$L=70.3mH,\ C=100\mu F$$



LEARNING EXTENSION

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



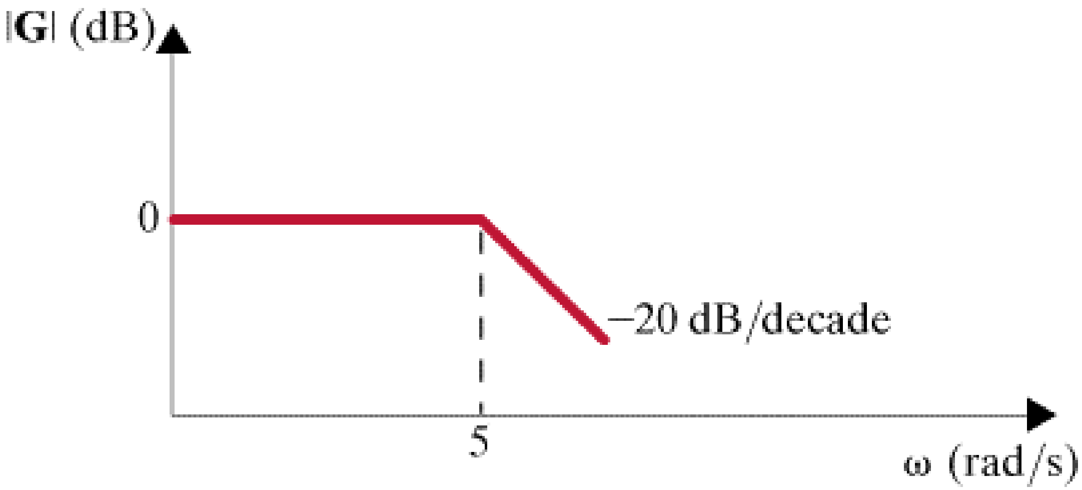
$$\tau = RC = (10\times10^{3}\Omega)(20\times10^{-6}F) = 0.2 rad/s$$

$$G_v(j\omega) = \frac{\dfrac{1}{j\omega C}}{R+\dfrac{1}{j\omega C}} = \frac{1}{\boxed{1+j\omega RC}}$$

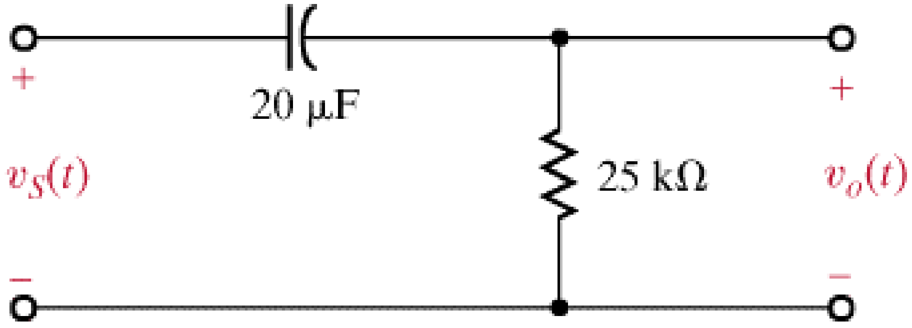
Break/corner frequency : 5rad/s

low frequency asymptote of 0dB/dec

High frequency asymptote of -20dB/dec



LEARNING EXTENSION Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



$$\tau = RC = (25\times10^{3}\Omega)(20\times10^{-6}F) = 0.5 rad/s$$

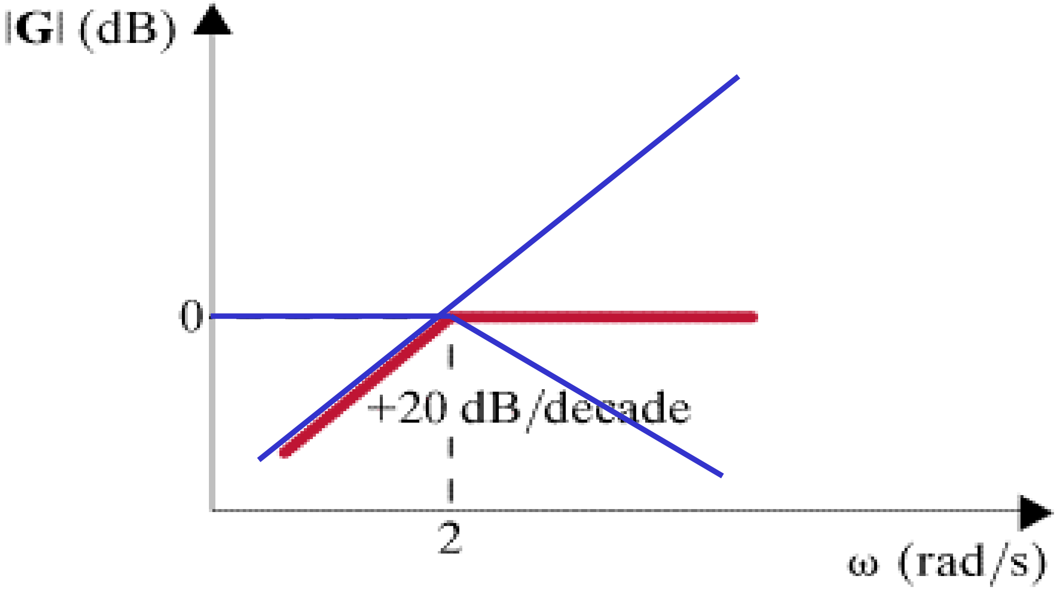
20dB/dec. Crosses 0dB at $\omega = \frac{1}{\tau} = 2 rad/s$

$$G_v(j\omega) = \frac{R}{R+\frac{1}{j\omega C}} = \frac{j\omega RC}{1+j\omega RC}$$

Break/corner frequency : 2rad/s

low frequency asymptote of 0dB/dec

High frequency asymptote of -20dB/dec



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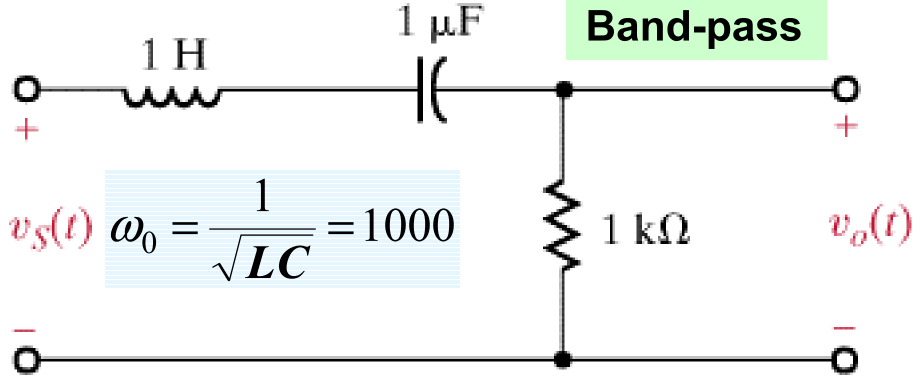
LEARNING EXTENSION

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$

Band-pass



$$\omega_0 = \frac{1}{\sqrt{LC}} = 1000$$

20dB/dec. Crosses 0dB at $\omega = \frac{1}{RC} = 1000\,rad/s$

$$G_v(j\omega) = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC}$$

$$\tau^2 = LC \Rightarrow \tau = \sqrt{10^{-6}} = 10^{-3},$$

$$2\varsigma\tau = RC = 10^3 \times 10^{-6} \Rightarrow \varsigma = \frac{10^{-3}}{2\times 10^{-3}} = 0.5$$

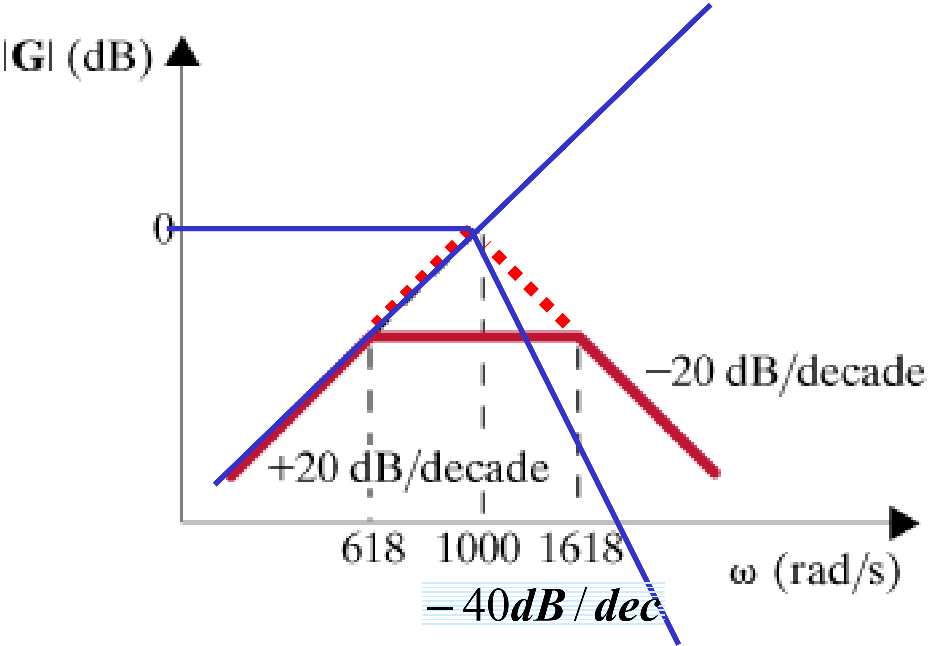
Break/corner frequency: 1000 rad/s

low frequency asymptote of 0dB/dec

High frequency asymptote of -40dB/dec

$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2} = 618\,rad/s$$

$$\omega_{HI} = \frac{(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2} = 1618\,rad/s$$



◀ ▶ GEAUX

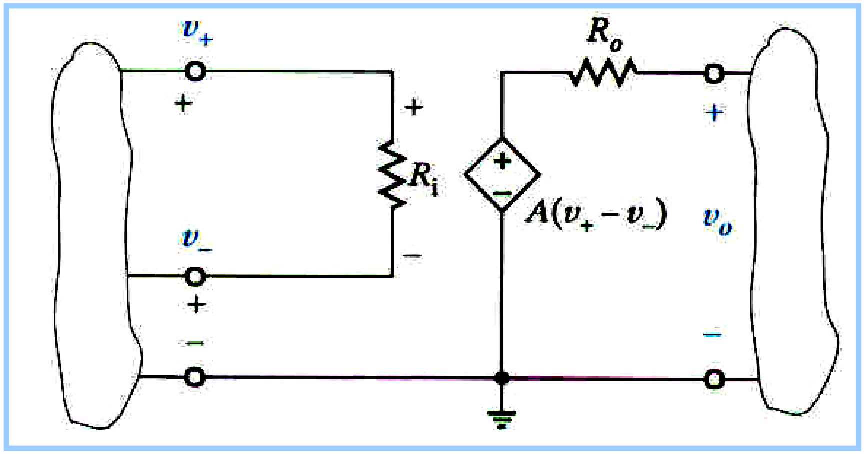
ACTIVE FILTERS

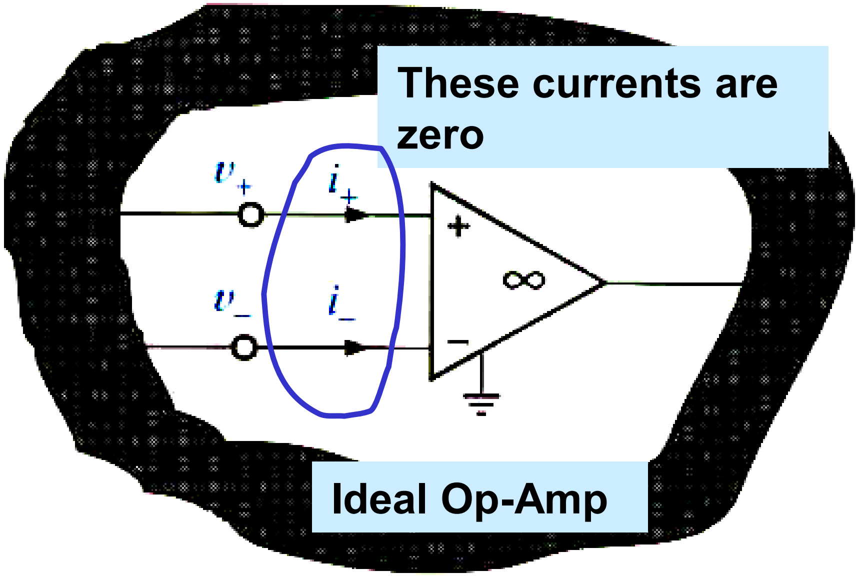
Passive filters have several limitations

1. Cannot generate gains greater than one
2. Loading effect makes them difficult to interconnect
3. Use of inductance makes them difficult to handle

Using operational amplifiers one can design all basic filters, and more, with only resistors and capacitors

The linear models developed for operational amplifiers circuits are valid, in a more general framework, if one replaces the resistors by impedances





These currents are zero

Ideal Op-Amp

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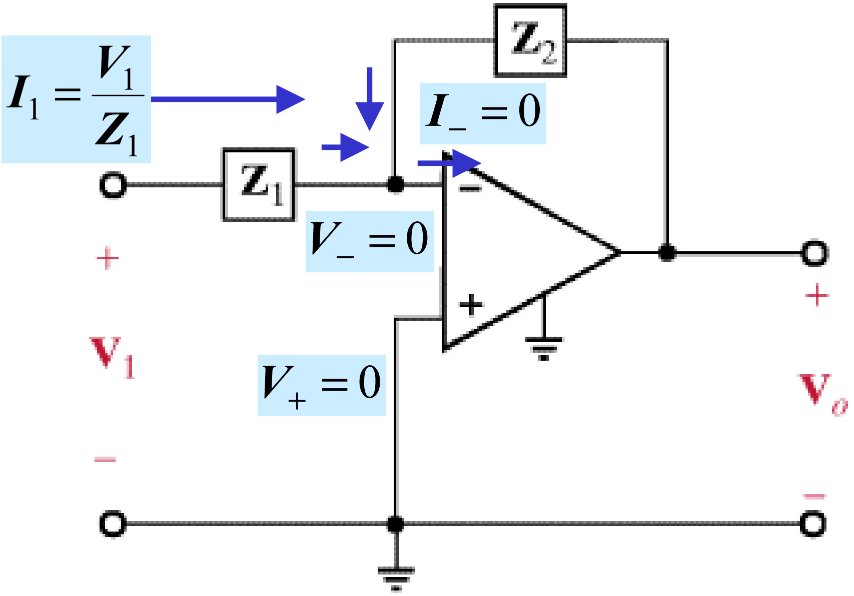
Basic Inverting Amplifier



$I_1 = \frac{V_1}{Z_1}$

$I_- = 0$

$V_- = 0$

$V_+ = 0$

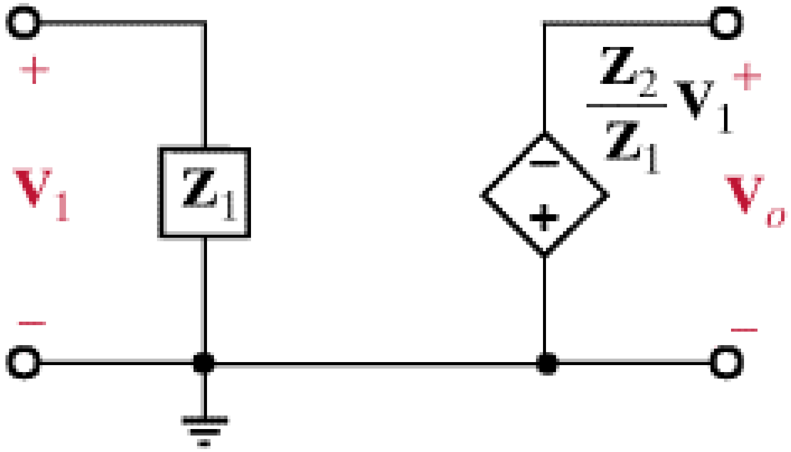
Infinite gain $\Rightarrow V_- = V_+$

Infinite input impedance $\Rightarrow I_- = I_+ = 0$

$$\frac{V_1}{Z_1} + \frac{V_2}{Z_2} = 0$$

$$V_2 = -\frac{Z_2}{Z_1} V_1$$

$$G = -\frac{Z_2}{Z_1}$$



Linear circuit equivalent

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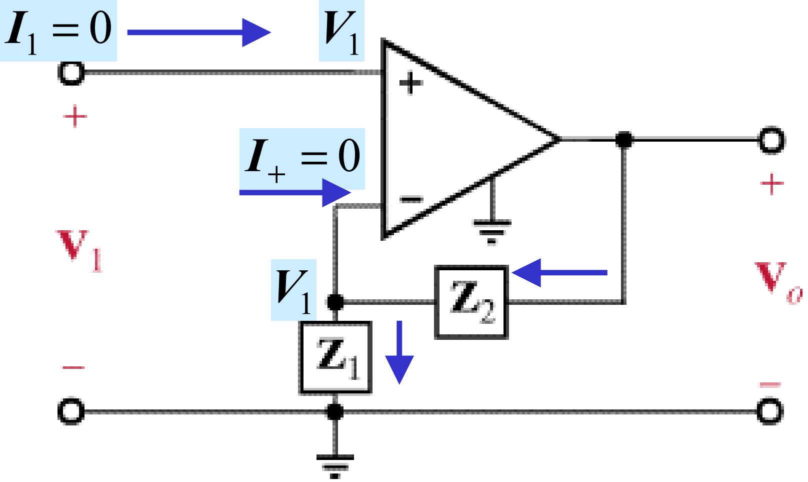
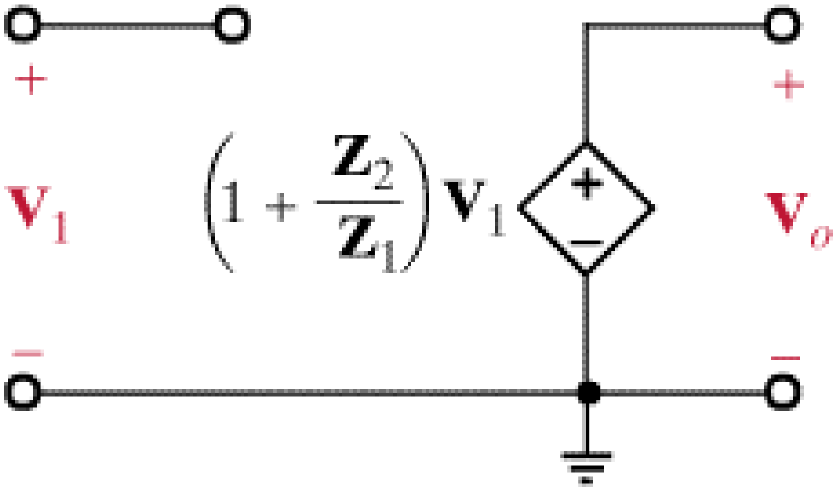
Basic Non-inverting amplifier



$$\frac{V_0 - V_1}{Z_2} = \frac{V_1}{Z_1}$$

$$V_0 = \frac{Z_2 + Z_1}{Z_1} V_1$$

$$G = 1 + \frac{Z_2}{Z_1}$$



Basic Non-inverting Amplifier

Due to the internal op-amp circuitry, it has limitations, e.g., for high frequency and/or low voltage situations. The *Operational Transductance Amplifier* (OTA) performs well in those situations

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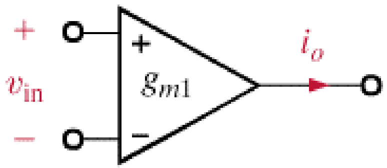
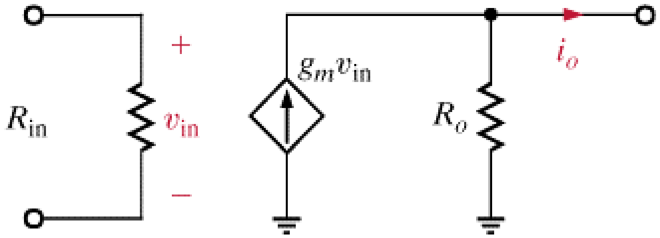
Operational Transductance Amplifier



(a)



(b)

$$\text{Ideal OTA}: \ R_{in} = R_0 = \infty$$

Comparison of Op-Amp and OTA

Amplifier Type	Ideal Rin	Ideal Ro	Ideal Gain	Input Current	input Voltage
Op-Amp	∞	0	∞	0	0
OTA	∞	∞	gm	0	nonzero

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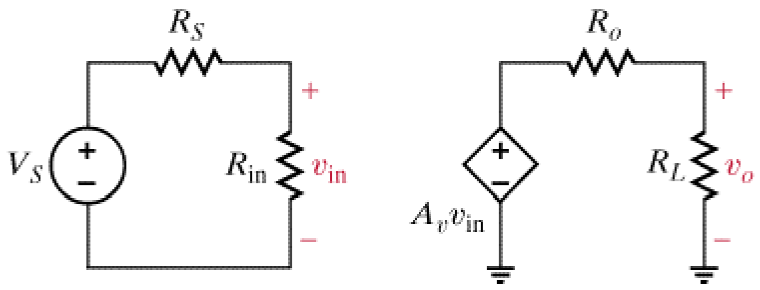


Basic Op-Amp Circuit

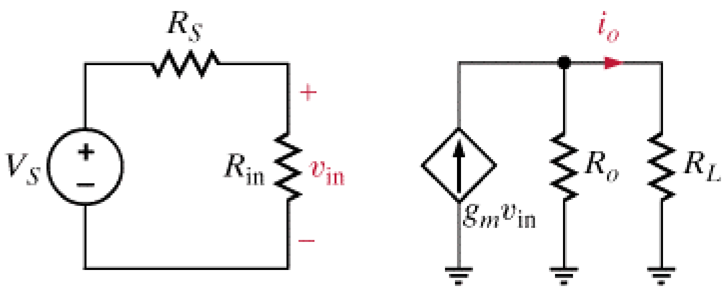
$$v_0 = \frac{R_L}{R_0 + R_L} A_v v_{in}$$

$$v_{in} = \frac{R_{in}}{R_S + R_{in}} V_S$$

$$A = \frac{v_0}{V_S} = \left[\frac{R_L}{R_0 + R_L}\right] A_v \left[\frac{R_{in}}{R_S + R_{in}}\right]$$

Ideal Op - Amp

$$A = A_v = \infty$$



Basic OTA Circuit

$$i_0 = \frac{R_0}{R_0 + R_L} g_m v_{in}$$

$$v_{in} = \frac{R_{in}}{R_S + R_{in}} V_S$$

$$G_m = \frac{i_0}{v_{in}} = \left[\frac{R_0}{R_0 + R_L}\right] g_m \left[\frac{R_{in}}{R_S + R_{in}}\right]$$

Ideal OTA

$$G_m = g_m$$

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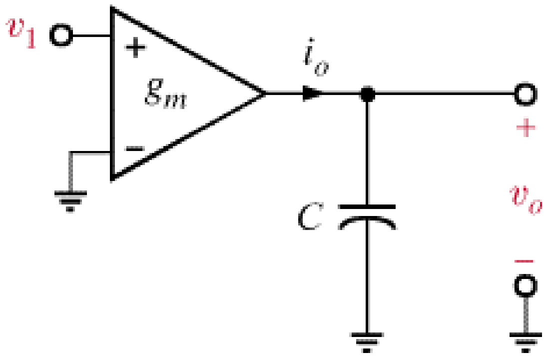
Basic OTA Circuits



$$i_0 = g_m v_1$$

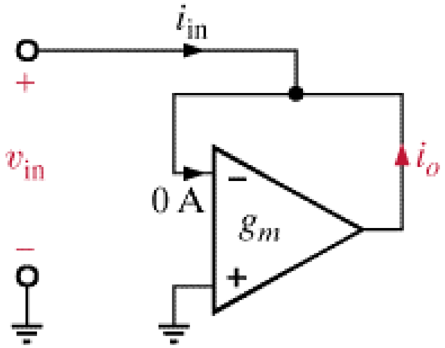
$$v_0 = \frac{1}{C}\int_0^t i_0(x)dx + v_0(0)$$

Integrator

$$v_0(t) = \frac{g_m}{C}\int_0^t v_1(x)dx + v_0(0)$$

In the frequency domain

$$V_0 = \frac{g_m}{j\omega C}V_1$$



$i_0 = -g_m v_{in}$ (notice polarity)

$i_{in} + i_0 = 0$

Simulated Resistor

$$\frac{v_{in}}{i_{in}} = R_{eq} = \frac{1}{g_m}$$

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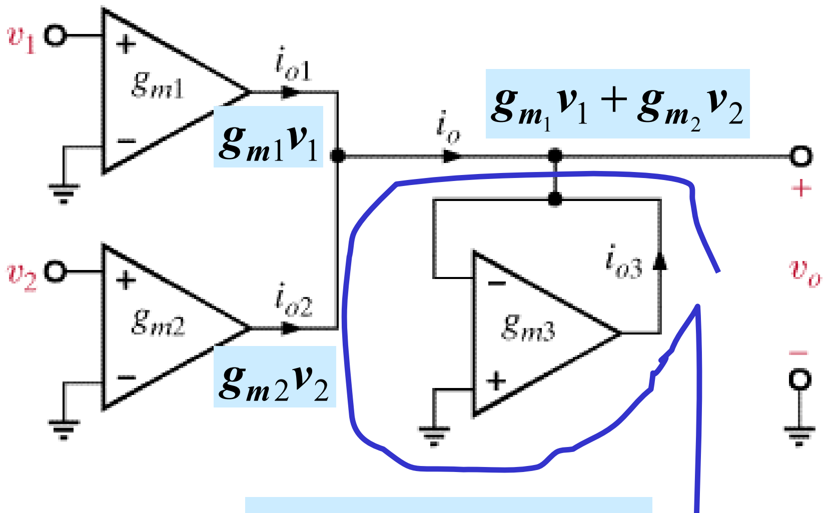


$g_{m1} v_1$

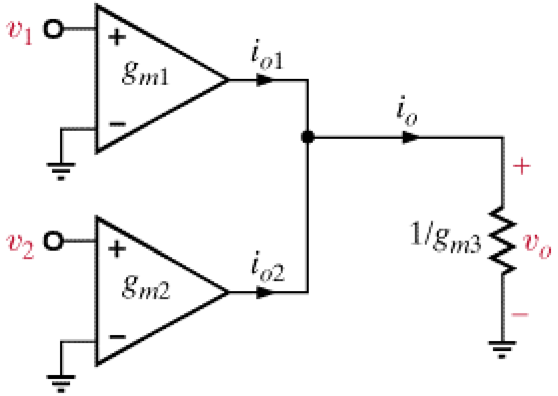
$g_{m2} v_2$

$g_{m_1} v_1 + g_{m_2} v_2$

Basic OTA Adder

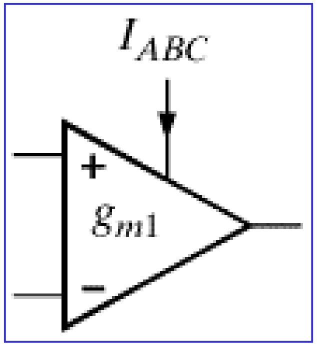
Simulated Resistor

$$v_0 = \frac{1}{g_{m3}} (g_{m_1} v_1 + g_{m_2} v_2)$$



Equivalent representation

Programmability of g_m



$$g_{m1} = \left[20 \frac{S}{A} \right] I_{ABC}$$

Typical values

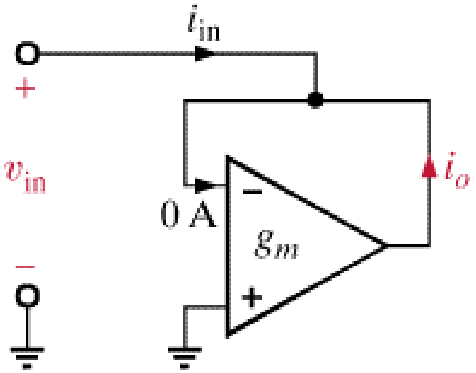
$g_m \le 10 mS$

g_m range: 3 - 7 decades

$(e.g., \ g_m \ge \frac{10 mS}{10^7})$

LEARNING EXAMPLE

Produce a $25k\Omega$ resistor



$$g_m \le 4mS$$

$$g_m \ge \frac{4mS}{10^4} = 4\times10^{-7}S$$

$$g_m = 20I_{ABC}$$

Simulated Resistor

$$\frac{v_{in}}{i_{in}} = R_{eq} = \frac{1}{g_m}$$

$$25\times10^3 = \frac{1}{g_m} \Rightarrow g_m = 4\times10^{-5}S > 4\times10^{-7}S$$

$$4\times10^{-5}S = 20\left[\frac{S}{A}\right]I_{ABC}(A)$$

$$I_{ABC} = 2\times10^{-6}A = 2\mu A$$

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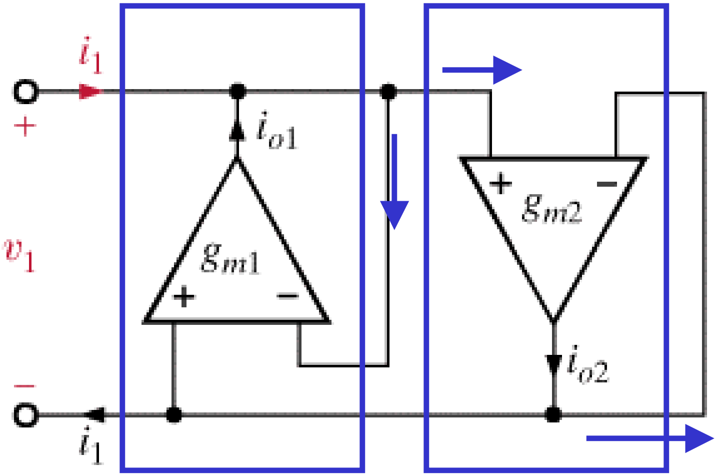
LEARNING EXAMPLE

Floating simulated resistor



$$i_{01} = -g_{m1} v_1$$

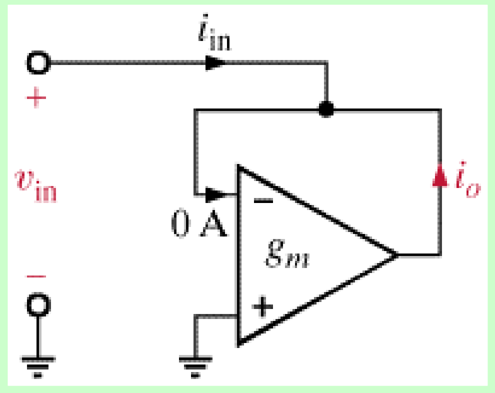
$$i_1 = -i_{01}$$

$$i_{02} = g_{m2} v_1$$

$$i_{02} = i_1$$

For proper operation

$$g_{m1} = g_{m2}$$



$$i_0 = -g_m v_{in}$$

One grounded terminal

Produce a 10MΩ resistor

$$g_m \le 4mS$$

$$g_m \ge \frac{4mS}{10^4} = 4 \times 10^{-7} S$$

$$g_m = 20 I_{ABC}$$

$$g_m = \frac{1}{10 \times 10^6} = 10^{-7} S$$

$$< 4 \times 10^{-7} S$$

The resistor cannot be produced with this OTA!

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LEARNING EXAMPLE

Select g_{m1}, g_{m2}, g_{m3}, to produce

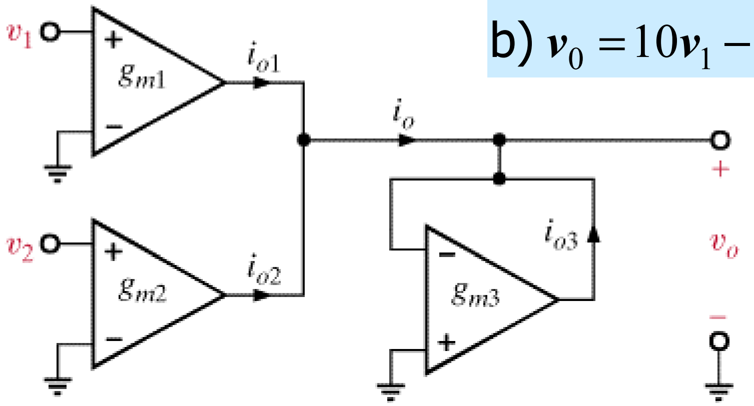
a) $v_0 = 10v_1 + 2v_2$

b) $v_0 = 10v_1 - 2v_2$

$$g_m \le 4mS$$

$$g_m \ge \frac{4mS}{10^4} = 4\times10^{-7} S$$

$$g_m = 20 I_{ABC}$$



$$v_0 = \frac{1}{g_{m3}}(g_{m_1}v_1 + g_{m_2}v_2)$$

Case a

$$\frac{g_{m1}}{g_{m3}} = 10; \quad \frac{g_{m2}}{g_{m3}} = 2$$

Two equations in three unknowns.
Select one transductance

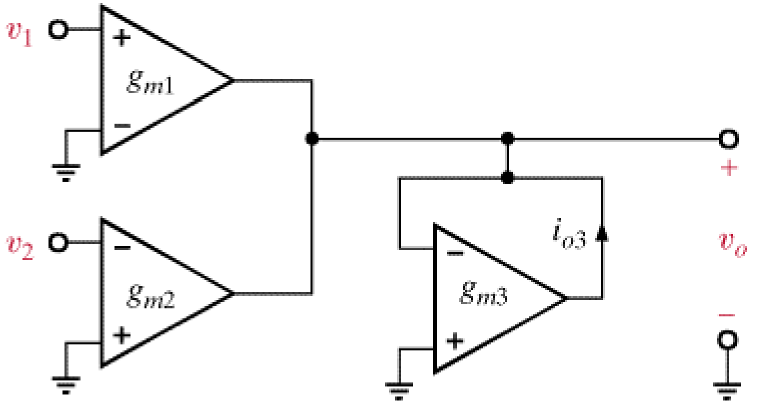
$$g_{m3} = 0.1mS \Rightarrow I_{ABC3} = \frac{1}{20}\times10^{-4}(A) = 5\mu A$$

$$g_{m2} = 0.2mS \Rightarrow I_{ABC2} = 10\mu A$$

$$g_{m1} = 1mS \Rightarrow I_{ABC1} = 50\mu A$$

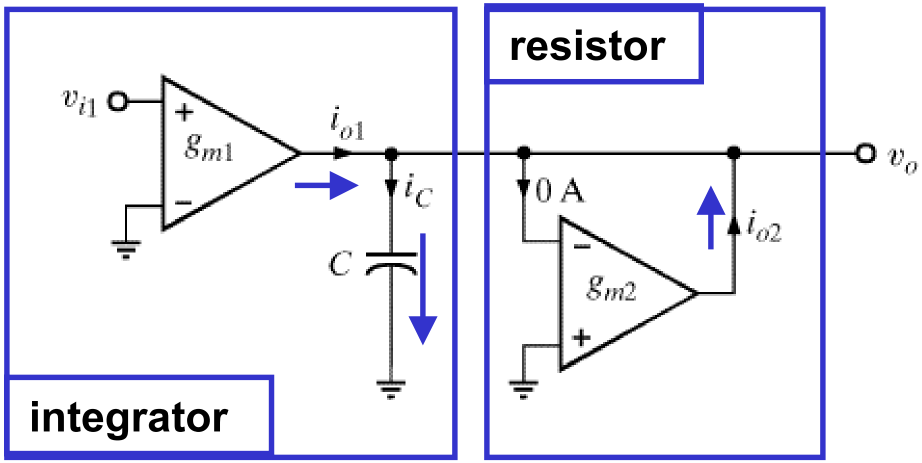
Case b

Reverse polarity of v2!



OTA-C CIRCUITS

Circuits created using capacitors, simulated resistors, adders and integrators



Frequency domain analysis assuming ideal OTAs

$$V_0 = \frac{1}{j\omega C} I_C$$

$$I_C = I_{01} + I_{02}$$

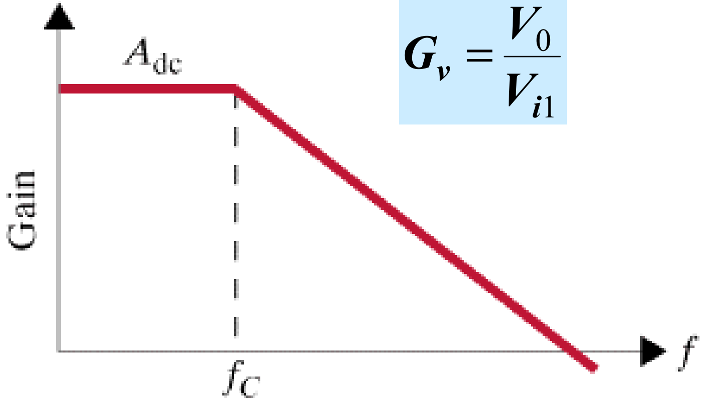
$$I_{01} = g_{m1} V_{i1}$$

$$I_{02} = -g_{m2} V_0$$

$$V_0 = \frac{1}{j\omega C}\left[g_{m1} V_{i1} - g_{m2} V_0\right]$$

$$V_0 = \frac{g_{m1}/g_{m2}}{1 + j\omega C/g_{m2}} V_{i1}$$

Magnitude Bode plot



$$G_v = \frac{V_0}{V_{i1}}$$

$$A_{dc} = \frac{g_{m1}}{g_{m2}}$$

$$\omega_C = \frac{g_{m2}}{C}$$

$$2\pi f_C = \frac{g_{m2}}{C}$$

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LEARNING EXAMPLE

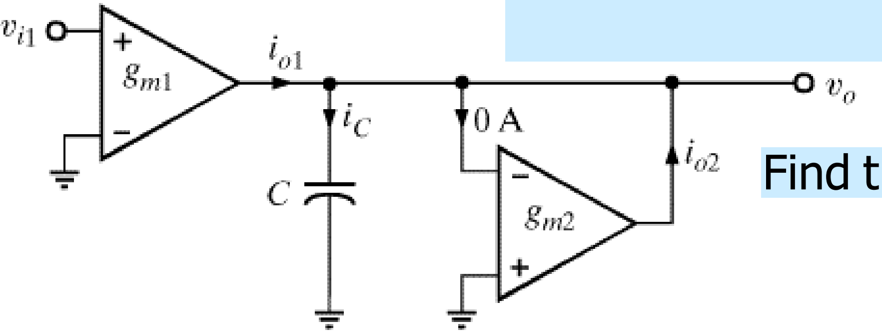
Desired: $G_v = \frac{V_0}{V_{i1}} = \frac{4}{1+\frac{j\omega}{2\pi(10^5)}}$

$g_m \le 1mS$

$g_m \ge \frac{1mS}{10^3} = 10^{-6} S$

OK

$g_m = 20 I_{ABC}$



Find the transductances and biases

$A_{dc} = \frac{g_{m1}}{g_{m2}}$

$\omega_C = \frac{g_{m2}}{C}$

$2\pi f_C = \frac{g_{m2}}{C}$

$A_{dc} = 4$

$\omega_C = 2\pi(10^5) \Rightarrow f_C = 100kHz$

Two equations in three unknowns.
Select the capacitor value

$C = 25pF$

$g_{m2} = 2\pi(10^5)(25\times10^{-12}) = 15.7\times10^{-6} S$

$I_{ABC2} = \frac{15.7}{20} = 0.785\mu A$

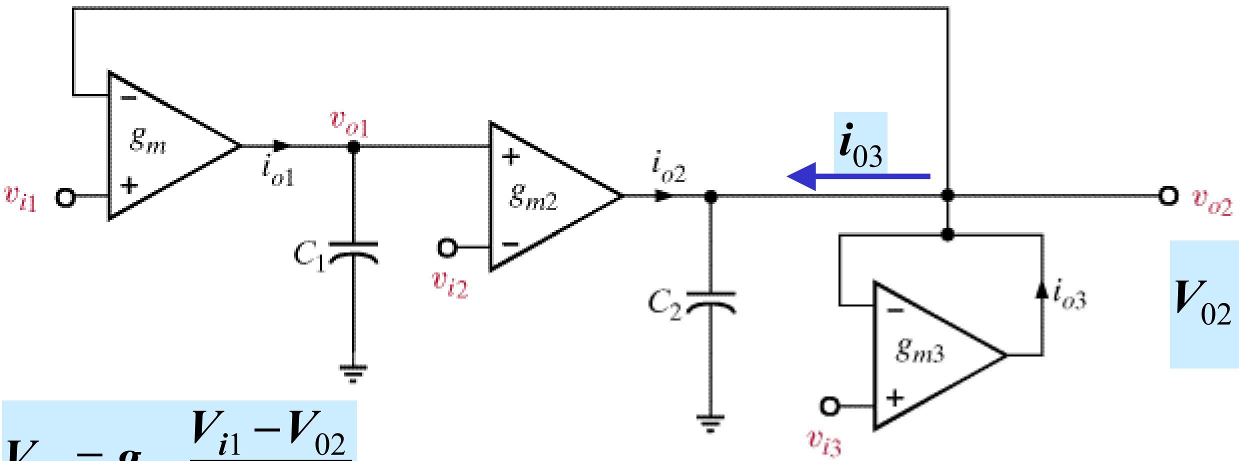
$g_{m1} = 62.8\mu S \Rightarrow I_{ABC1} = 3.14\mu A$

GEAUX

TOW-THOMAS OTA-C BIQUAD FILTER

biquad ~ biquadratic

$$\frac{V_0}{V_i} = \frac{A(j\omega)^2 + B(j\omega) + C}{\boxed{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + \omega_0^2}}$$



$$V_{02} = \frac{1}{j\omega C_2}(I_{02} + I_{03})$$

$$V_{01} = g_{m1}\frac{V_{i1} - V_{02}}{j\omega C}$$

$$I_{02} = g_{m2}(V_{01} - V_{i2})$$

$$I_{03} = g_{m3}(V_{i3} - V_{o2})$$

Four equations and four unknowns $(V_{01}, V_{02}, I_{01}, I_{02})$

$$V_{01} = \frac{\left[\frac{j\omega C_2}{g_{m2}} + \frac{g_{m3}}{g_{m2}}\right]V_{i1} + V_{i2} - \left[\frac{g_{m3}}{g_{m2}}\right]V_{i3}}{\boxed{\left[\frac{C_1C_2}{g_{m1}g_{m2}}\right](j\omega)^{2+}\left[\frac{g_{m3}C_1}{g_{m2}g_{m1}}\right](j\omega) + 1}}$$

$$V_{02} = \frac{V_{i1} - \left[\frac{j\omega C_1}{g_{m1}}\right]V_{i2} + \left[\frac{j\omega C_1 g_{m3}}{g_{m1}g_{m2}}\right]V_{i3}}{\left[\frac{C_1C_2}{g_{m1}g_{m2}}\right](j\omega)^{2+}\left[\frac{g_{m3}C_1}{g_{m2}g_{m1}}\right](j\omega) + 1}$$

$$\omega_0 = \sqrt{\frac{g_{m1}g_{m2}}{C_1C_2}},\quad \frac{\omega_0}{Q} = \frac{g_{m3}}{C_2},\quad Q = \sqrt{\frac{g_{m1}g_{m2}}{g_{m3}^2}}\sqrt{\frac{C_2}{C_1}}$$

Filter Type	A	B	C
Low-pass	0	0	nonzero
Band-pass	0	nonzero	0
High-pass	nonzero	0	0

$$\left.\begin{matrix} g_{m1} = g_{m2} \\ C_1 = C_2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \omega_0 = \sqrt{g_m/C} \\ Q = \dfrac{g_m}{g_{m3}} \\ BW = \dfrac{g_{m3}}{C} \end{matrix}\right.$$

GEAUX

LEARNING EXAMPLE

Design a band-pass filter with center frequency of 500kHz, bandwidth of 75kHz, and center frequency gain -5.

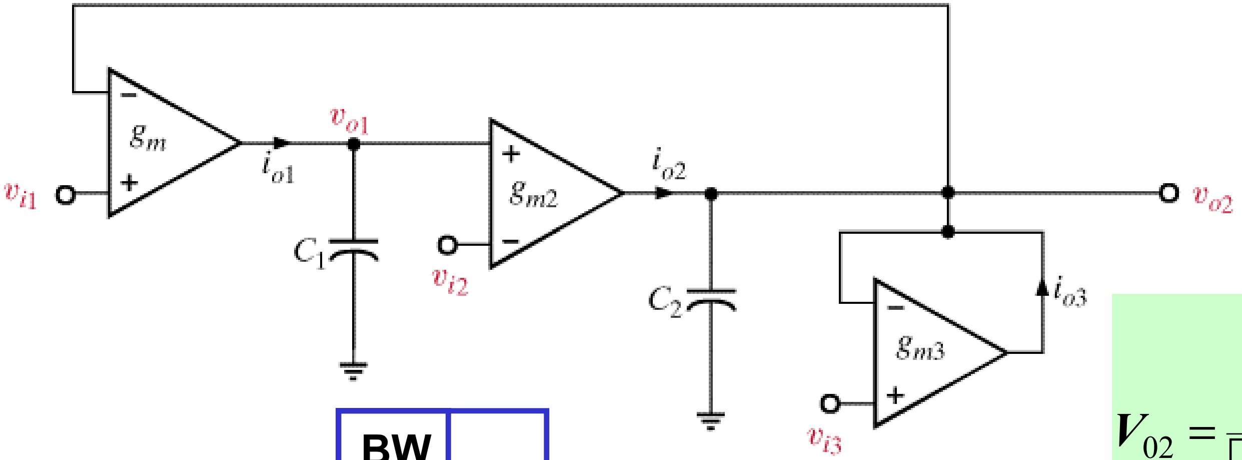
Use the Tow-Thomas configuration and 50-pF capacitors

$C_1 = C_2$

$g_m \le 4mS$

$g_m \ge \frac{4mS}{10^4} = 4\times10^{-7}\,S$

$g_m = 20 I_{ABC}$ $\quad V_{i3} = 0$



$$V_{02} = \frac{V_{i1} - \left[\frac{j\omega C_1}{g_{m1}}\right]V_{i2} + \left[\frac{j\omega C_1 g_{m3}}{g_{m1}g_{m2}}\right]V_{i3}}{\left[\frac{C_1C_2}{g_{m1}g_{m2}}\right](j\omega)^{2+}\left[\frac{g_{m3}C_1}{g_{m2}g_{m1}}\right](j\omega)+1}$$

BW

$$\omega_0 = \sqrt{\frac{g_{m1}g_{m2}}{C_1C_2}},\ \frac{\omega_0}{Q} = \frac{g_{m3}}{C_2},\ Q = \sqrt{\frac{g_{m1}g_{m2}}{g_{m3}^2}}\sqrt{\frac{C_2}{C_1}}$$

$$\omega = \omega_0 \Rightarrow 1 + \left[\frac{C_1C_2}{g_{m1}g_{m2}}\right](j\omega)^2 = 0$$

$$BW = \frac{g_{m3}}{50\times10^{-12}} = 2\pi\times75\times10^3 = 23.56\mu S$$

$$|G_v(j\omega_0)| = \frac{g_{m2}}{g_{m3}} = 5 \Rightarrow g_{m2} = 117.8\mu S$$

$$(\omega_0)^2 = \left(2\pi\times5\times10^5\right)^2 = \frac{g_{m1}\times117.8\times10^{-6}}{(5\times10^{-13})^2} \Rightarrow g_{m1} = 209.5\mu S$$

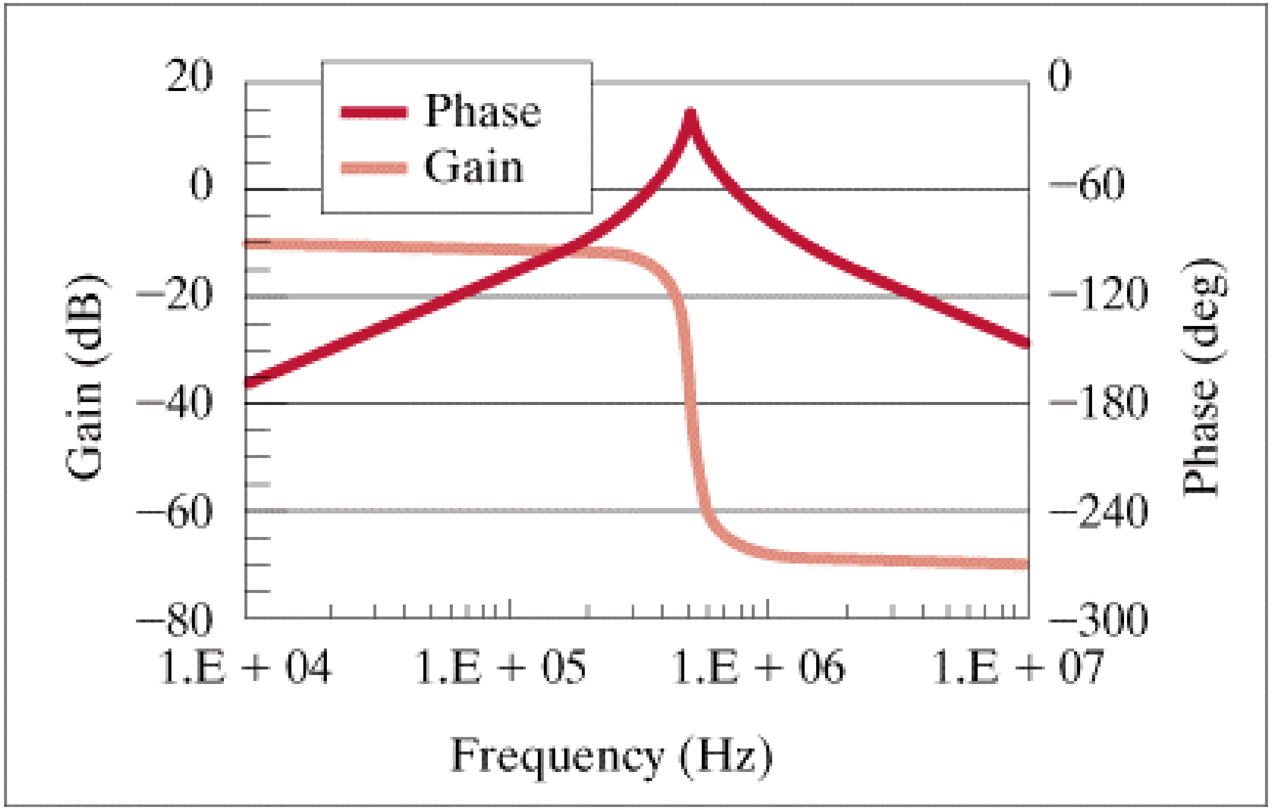
$I_{ABC1} = 10.47\mu A$

$I_{ABC2} = 5.89\mu A$

$I_{ABC3} = 1.18\mu A$

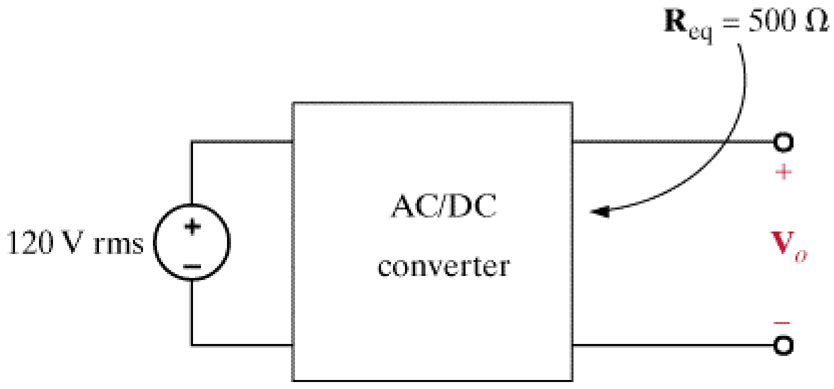
GEAUX

Bode plots for resulting amplifier

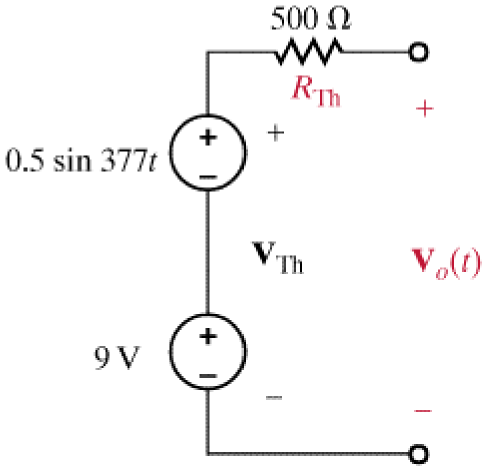


LEARNING BY APPLICATION

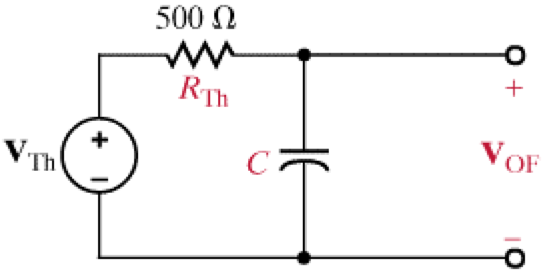
Using a low-pass filter to reduce 60Hz ripple



(a)



Thevenin equivalent for AC/DC converter



Using a capacitor to create a low-pass filter

$$V_{OF} = \frac{1}{1 + j\omega R_{TH} C} V_{TH}$$

$$\omega_C = \frac{1}{R_{TH} C}$$

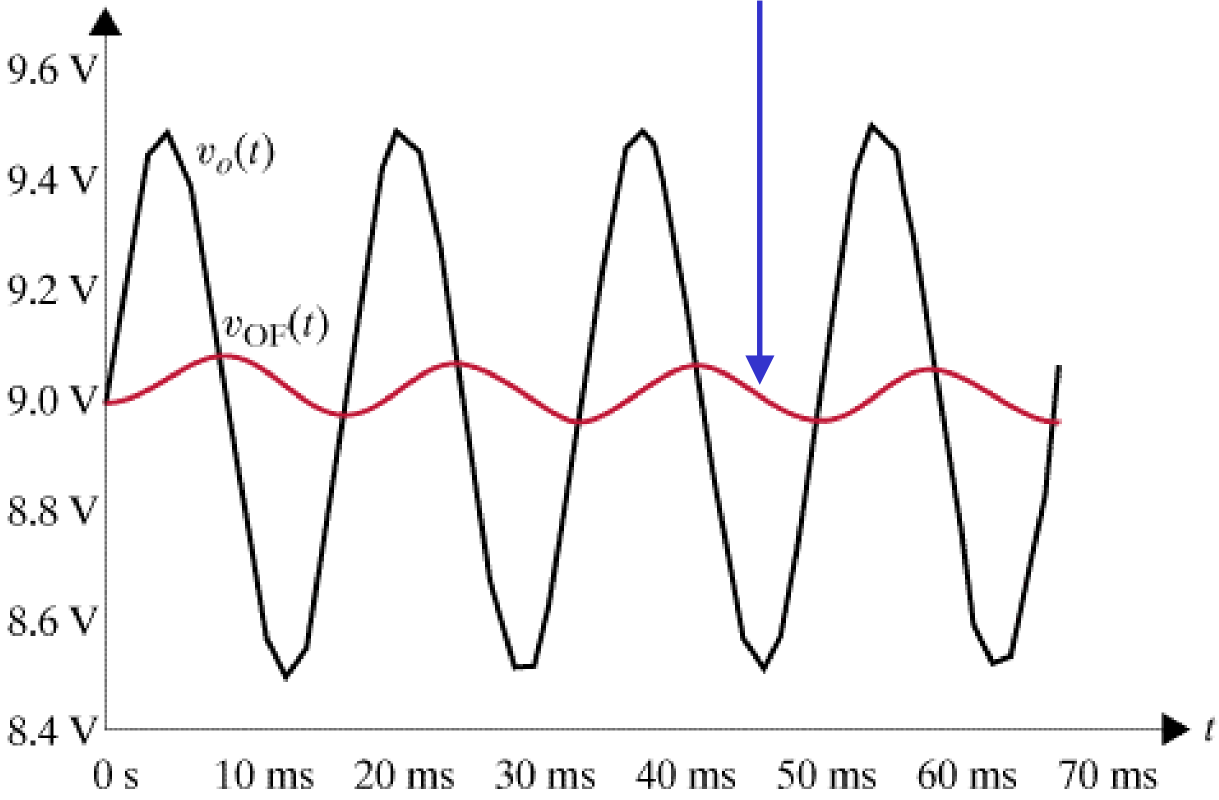
$$|V_{OF}| = \frac{|V_{TH}|}{\sqrt{1 + (\omega R_{TH} C)^2}}$$

Design criterion: place the corner frequency at least a decade lower

$$|V_{OF}| \approx 0.1 |V_{TH}|$$

$$500C = \frac{1}{2\pi \times 6} \Rightarrow C = 53.05 \mu F$$

GEAUX



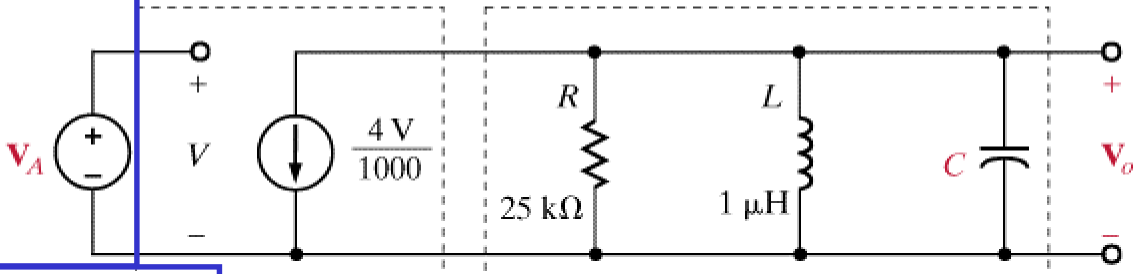
(d)

Filtered output

LEARNING EXAMPLE

Single stage tuned transistor amplifier

Select the capacitor for maximum gain at 91.1MHz



Antenna Voltage

Transistor

Parallel resonant circuit

(a)

$$\frac{V_0}{V_A} = -\frac{4}{1000}\left[R \,\|\, j\omega L \,\|\, {}^{1}\!/_{j\omega C}\right]$$

$$= -\frac{4}{1000}\frac{1}{\frac{1}{R}+\frac{1}{j\omega L}+j\omega C}\times\frac{j\omega/C}{j\omega/C}$$

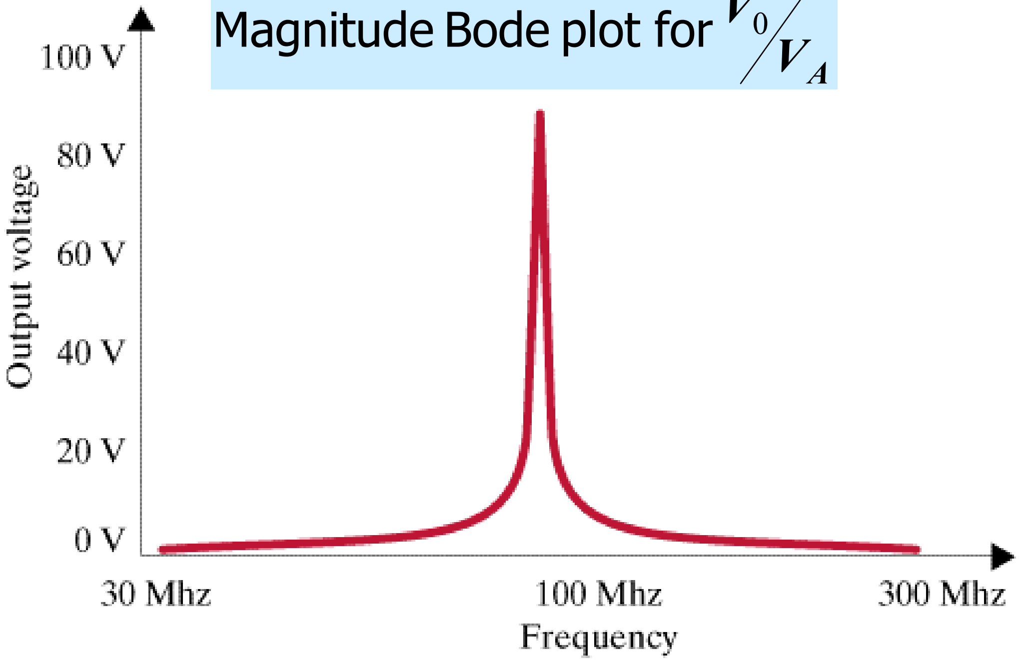
$$\frac{V_0}{V_A} = -\frac{4}{1000}\frac{j\omega/C}{(j\omega)^2+\frac{j\omega}{RC}+\frac{1}{LC}}$$

Band-pass with center frequency $1/\sqrt{LC}$

$$2\pi\left(91.1\times10^6\right)=\frac{1}{\sqrt{10^{-6}C}}\Rightarrow \quad C = 3.05\,pF$$

$$\left|\frac{V_0}{V_A}\right|\left(\omega=\frac{1}{\sqrt{LC}}\right)=\frac{4}{1000}R=100$$

Magnitude Bode plot for ${}^{V_0}\!/_{V_A}$



(b)

GEAUX

LEARNING BY DESIGN

Anti-aliasing filter

Nyquist Criterion

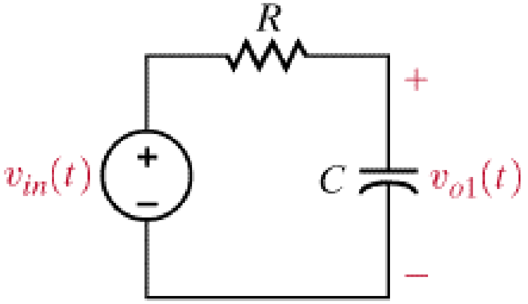
When digitizing an analog signal, such as music, any frequency components greater than half the sampling rate will be distorted

In fact they may appear as spurious components. The phenomenon is known as aliasing.

SOLUTION: Filter the signal before digitizing, and remove all components higher than half the sampling rate. Such a filter is an anti-aliasing filter

For CD recording the industry standard is to sample at 44.1kHz.
An anti-aliasing filter will be a low-pass with cutoff frequency of 22.05kHz

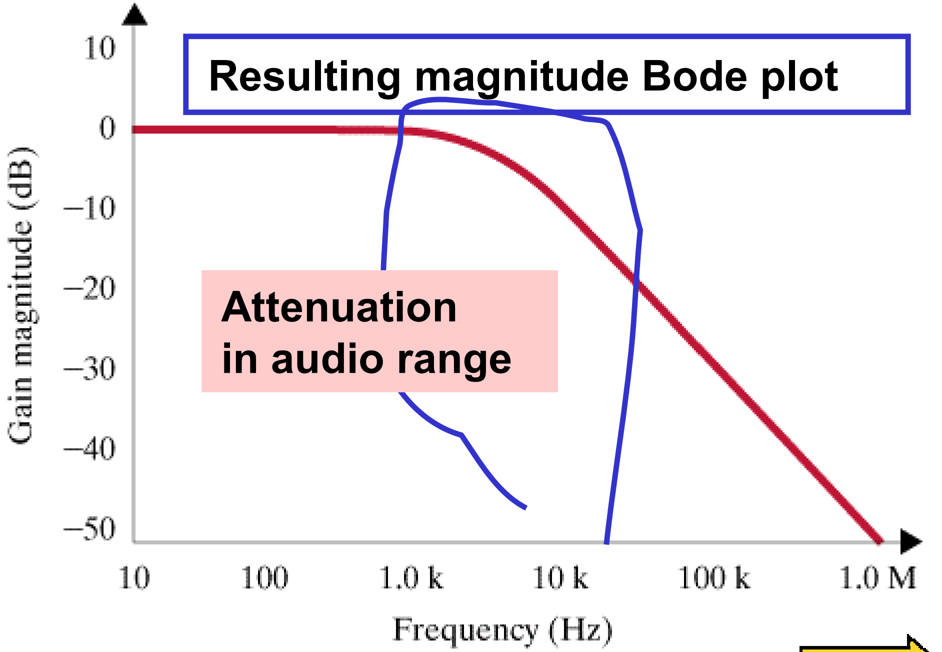
Single-pole low-pass filter



$$\frac{V_{01}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

$$\omega_C = \frac{1}{RC} = 2\pi \times 22{,}050$$

$$C = 1nF \Rightarrow R = 72.18k\Omega$$



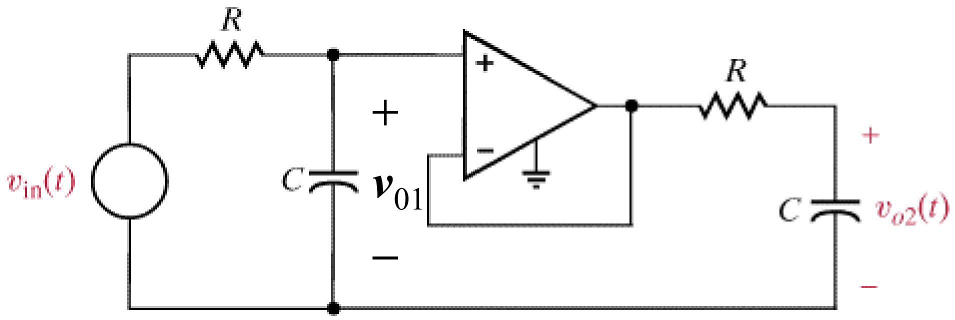
Resulting magnitude Bode plot

Attenuation in audio range

GEAUX

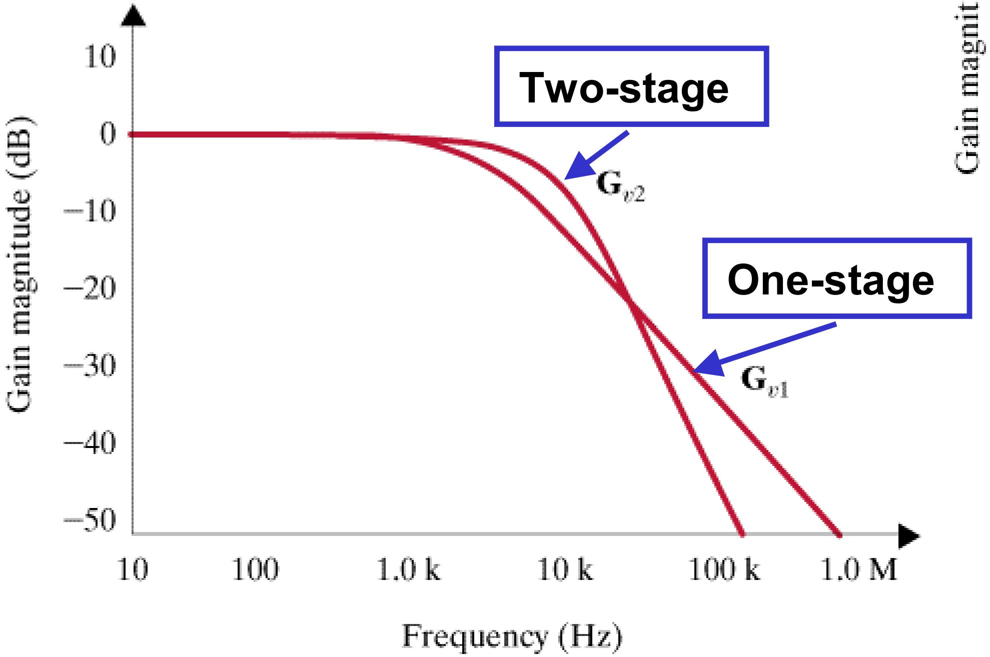
Improved anti-aliasing filter

Two-stage buffered filter



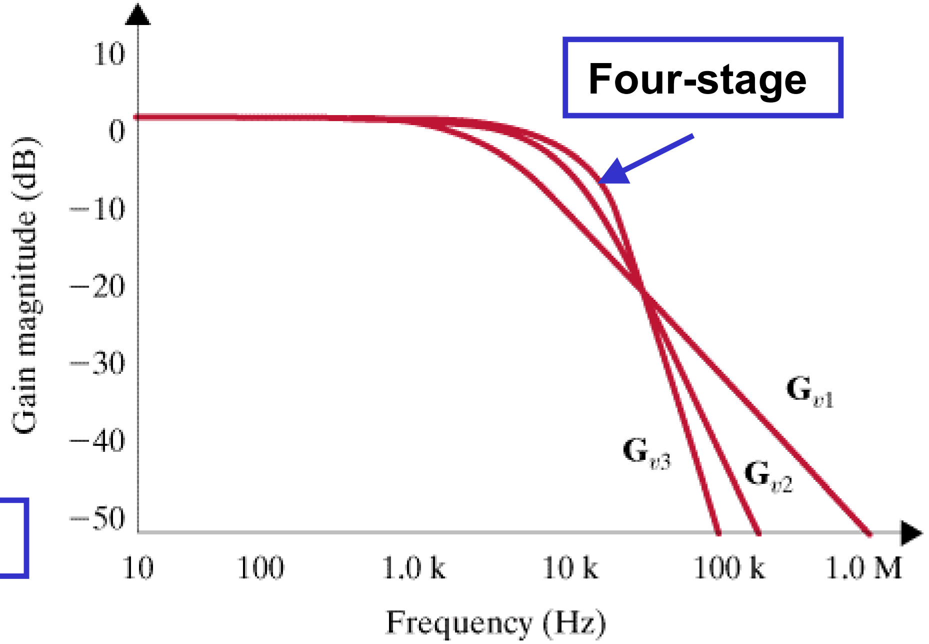
$$\frac{V_{01}}{V_{in}} = \frac{1}{1+j\omega RC}$$

$$\frac{V_{02}}{V_{01}} = \frac{1}{1+j\omega RC}$$



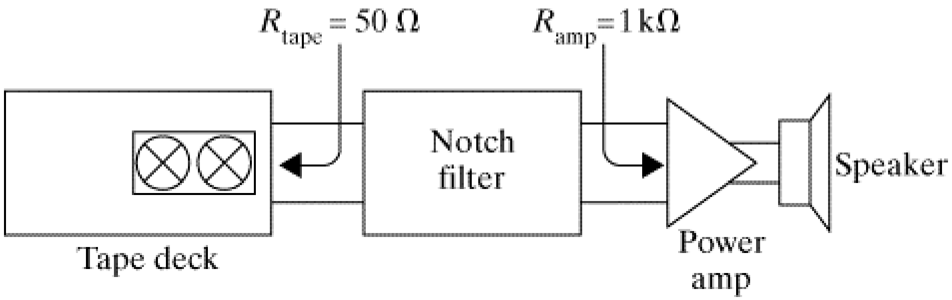
n - stage

$$\frac{V_{0n}}{V_{in}} = \frac{1}{(1+j\omega RC)^n}$$

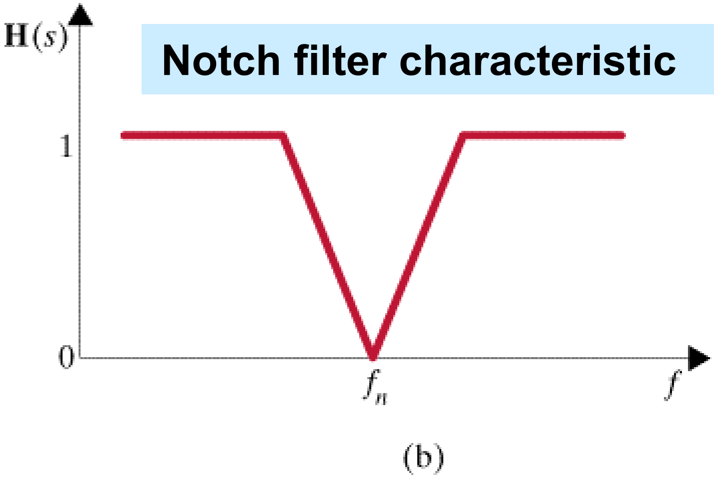


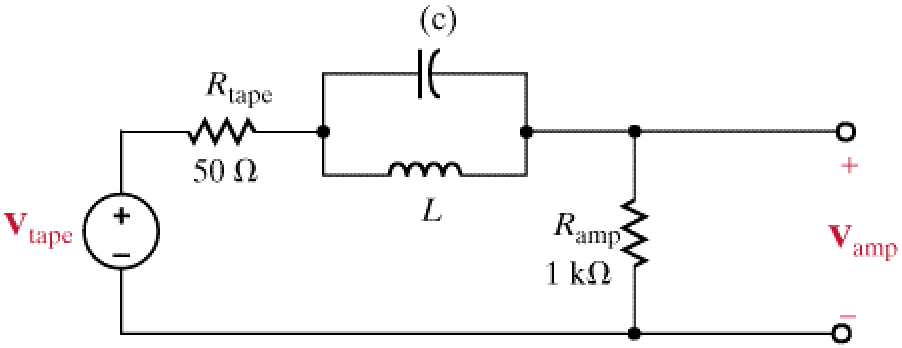
LEARNING BY DESIGN

Notch filter to eliminate 60Hz hum



Notch filter characteristic

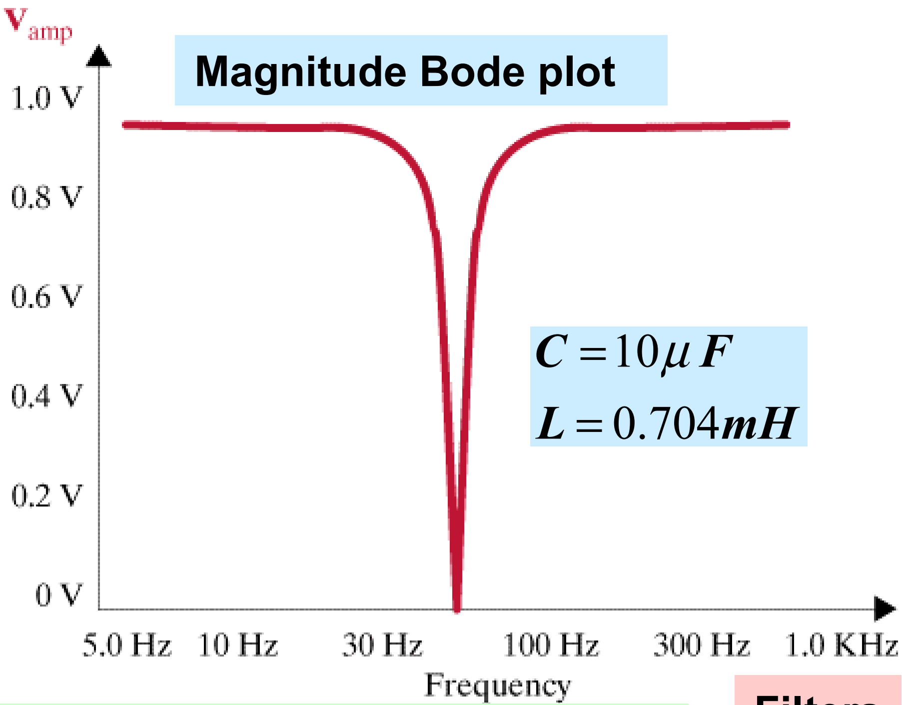




$$\frac{V_{amp}}{V_{tape}} = \frac{R_{amp}}{R_{amp} + R_{tape} + \left(sL \,\|\, 1/sC\right)}$$

$$\frac{V_{amp}}{V_{tape}} = \frac{R_{amp}}{R_{amp} + R_{tape}}\left[\frac{s^2LC + 1}{s^2LC + s\left(\frac{L}{R_{amp} + R_{tape}}\right) + 1}\right]$$

Magnitude Bode plot



$C = 10\mu F$

$L = 0.704mH$

notch frequency $= \dfrac{1}{\sqrt{LC}}$

To design, pick one, e.g., C and determine the other

Filters