# ANALYSIS OF CIRCUITS WITH ONE ENERGY STORING ELEMENT
## CONSTANT INDEPENDENT SOURCES

## A STEP-BY-STEP APPROACH

THIS APPROACH RELIES ON THE KNOWN FORM OF THE SOLUTION BUT FINDS THE CONSTANTS $K_1, K_2, \tau$ USING BASIC CIRCUIT ANALYSIS TOOLS AND FORGOES THE DETERMINATION OF THE DIFFERENTIAL EQUATION MODEL

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$K_1$ is the steady state value of the variable and can be determined analyzing the circuit in steady state

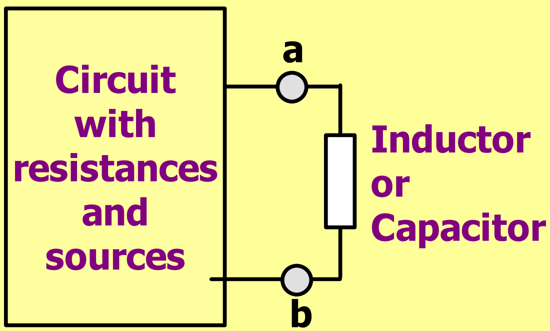
$x(0+)$ is the initial condition and provides the second equation to compute the constants $K_1, K_2$

$\tau$ is the time constant and can be determined using Thevenin across the energy storing element

GEAUX

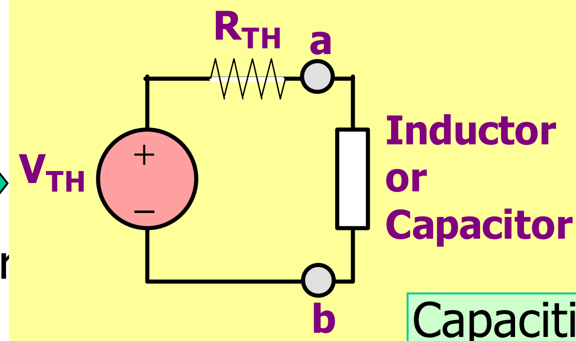
# CIRCUITS WITH ONE ENERGY STORING ELEMENT

## Obtaining the time constant: A General Approach



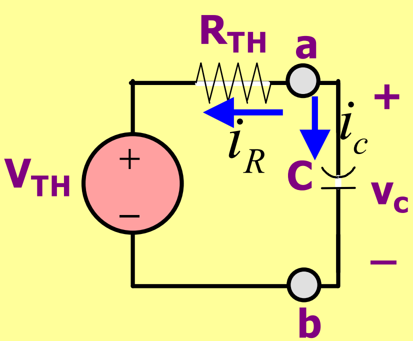

Representation of an arbitrary circuit with one storage element

Thevenin




Capacitive Circuit $\tau = R_{TH}C$

Inductive Circuit $\tau = \dfrac{L}{R_{TH}}$




**Case 1.1**
**Voltage across capacitor**

KCL@ node a

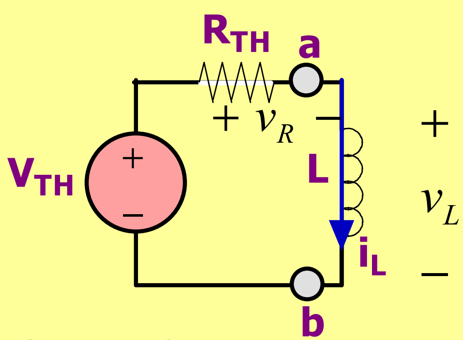
$$i_c + i_R = 0$$

$$i_c = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_C - v_{TH}}{R_{TH}}$$

$$C\frac{dv_C}{dt} + \frac{v_C - v_{TH}}{R_{TH}} = 0$$

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$




**Case 1.2**
**Current through inductor**

Use KVL

$$v_R + v_L = v_{TH}$$

$$v_R = R_{TH}i_L$$

$$v_L = L\frac{di_L}{dt}$$

$$L\frac{di_L}{dt} + R_{TH}i_L = v_{TH}$$

$$\left(\frac{L}{R_{TH}}\right)\frac{di_L}{dt} + i_L = \frac{v_{TH}}{R_{TH}} = i_{SC}$$

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## THE STEPS

STEP 1. THE FORM OF THE SOLUTION

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$$K_1 = x(\infty);\ K_1 + K_2 = x(0+)$$

DETERMINE $x(0+)$

STEP 2: DRAW THE CIRCUIT IN STEADY STATE PRIOR TO THE SWITCHING AND DETERMINE CAPACITOR VOLTAGE OR INDUCTOR CURRENT

STEP 3: DRAW THE CIRCUIT AT 0+ THE CAPACITOR ACTS AS A VOLTAGE SOURCE. THE INDUCTOR ACTS AS A CURRENT SOURCE.
DETERMINE THE VARIABLE AT t=0+

DETERMINE $x(\infty)$

STEP 4: DRAW THE CIRCUIT IN STEADY STATE AFTER THE SWITCHING AND DETERMINE THE VARIABLE IN STEADY STATE.

STEP 5: DETERMINE THE TIME CONSTANT

$\tau = R_{TH} C$ circuit with one capacitor

$\tau = \dfrac{L}{R_{TH}}$ circuit with one inductor

STEP 6: DETERMINE THE CONSTANTS K1, K2

$$K_1 = x(\infty),\quad K_1 + K_2 = x(0+)$$

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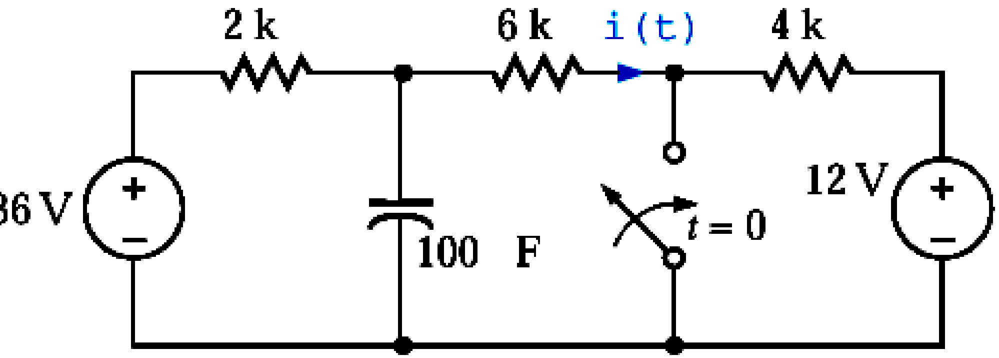
LEARNING EXAMPLE

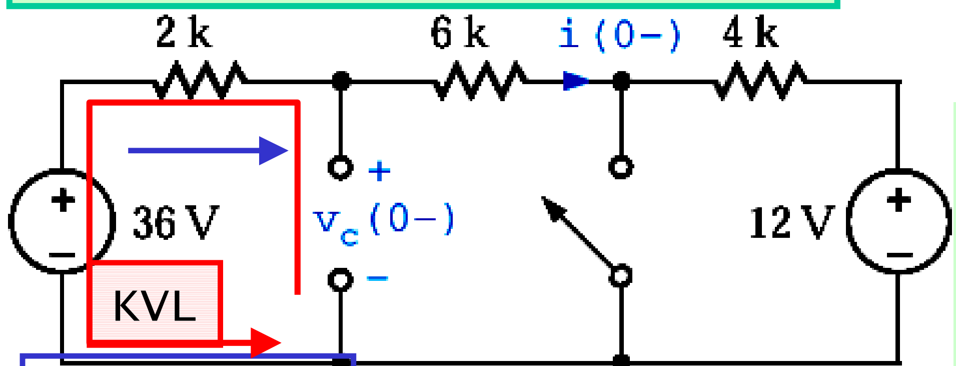
FIND $i(t), t > 0$




STEP 1: $i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: Initial voltage across capacitor

USE CIRCUIT IN STEADY STATE PRIOR TO THE SWITCHING



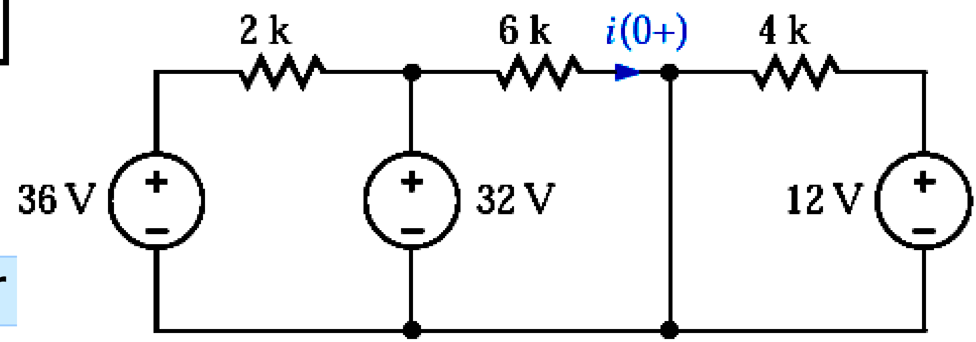

$i = \frac{24V}{12k\Omega} = 2mA$ $= i(0-)$

$v_c(0-) = 36V - (2mA)(2k) = 32[V]$

$v_C(0+) = v_c(0-)$

STEP 3: Determine $i(0+)$

USE A CIRCUIT VALID FOR t=0+. THE CAPACITOR ACTS AS SOURCE




$i(0+) = \frac{32V}{6k} = \frac{16}{3} mA$

NOTES FOR INDUCTIVE CIRCUIT

(1) DETERMINE INITIAL INDUCTOR CURRENT IN STEP 2

(2) FOR THE t=0+ CIRCUIT REPLACE INDUCTOR BY A CURRENT SOURCE

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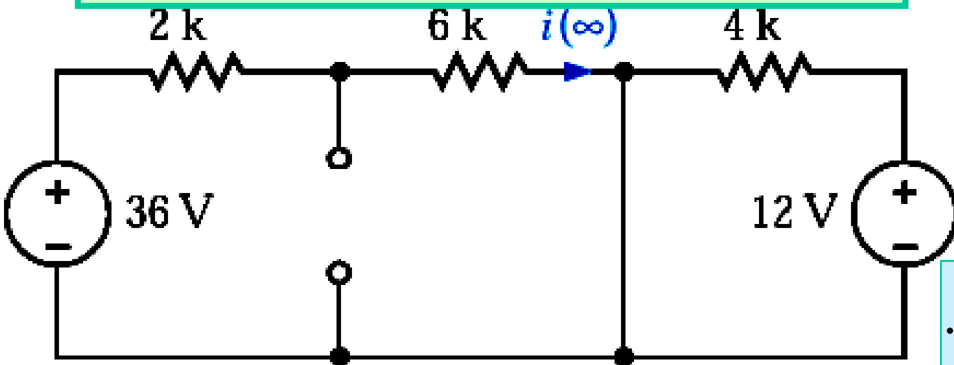
## STEP 4: Determine $i(\infty)$

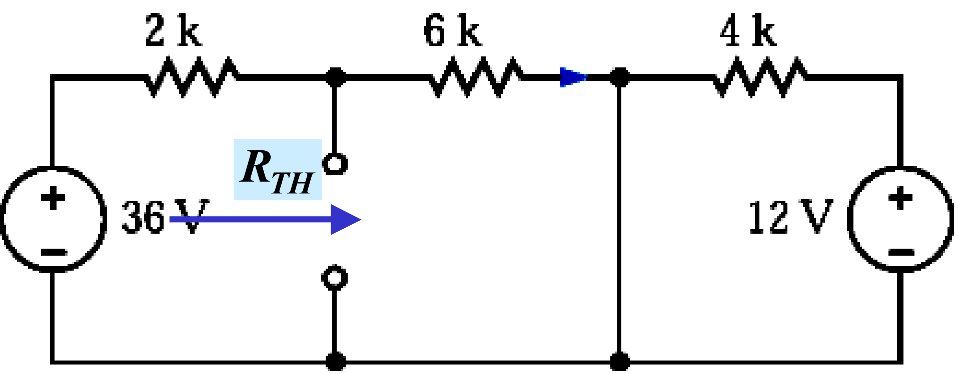
USE CIRCUIT IN STEADY STATE AFTER SWITCHING




$$i(\infty) = \frac{36}{8} mA$$

## STEP 5: Determine time constant

Capacitive circuit: $\tau = R_{TH} C$




$R_{TH} = 2k \,||\, 6k = 1.5k\Omega$ $\quad C = 100 \mu F$

$$\tau = (1.5 \times 10^{3} \Omega)(100 \times 10^{-6} F) = 0.15 s$$

## STEP 6: Determine $K_1, K_2$

(STEP 1) $\quad i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

(STEP 3) $\quad i(0+) = \frac{16}{3} mA$ $\quad = K_1 + K_2$

(STEP 4) $\quad i(\infty) = \frac{36}{8} mA$ $\quad = K_1$

$$\therefore K_2 = \frac{16}{3} - \frac{36}{8} = \frac{5}{6}$$

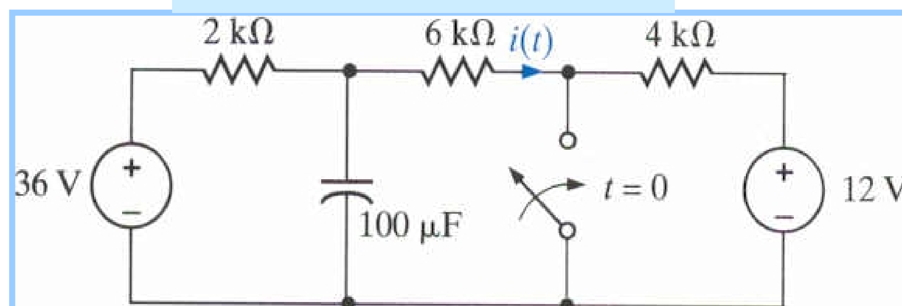
FINAL ANSWER

$$i(t) = \frac{36}{8} + \frac{5}{6} e^{-\frac{t}{0.15}}, t > 0$$

NOTE: FOR INDUCTIVE CIRCUIT

$$\tau = \frac{L}{R_{TH}}$$

ORIGINAL CIRCUIT

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# USING MATLAB TO DISPLAY FINAL ANSWER

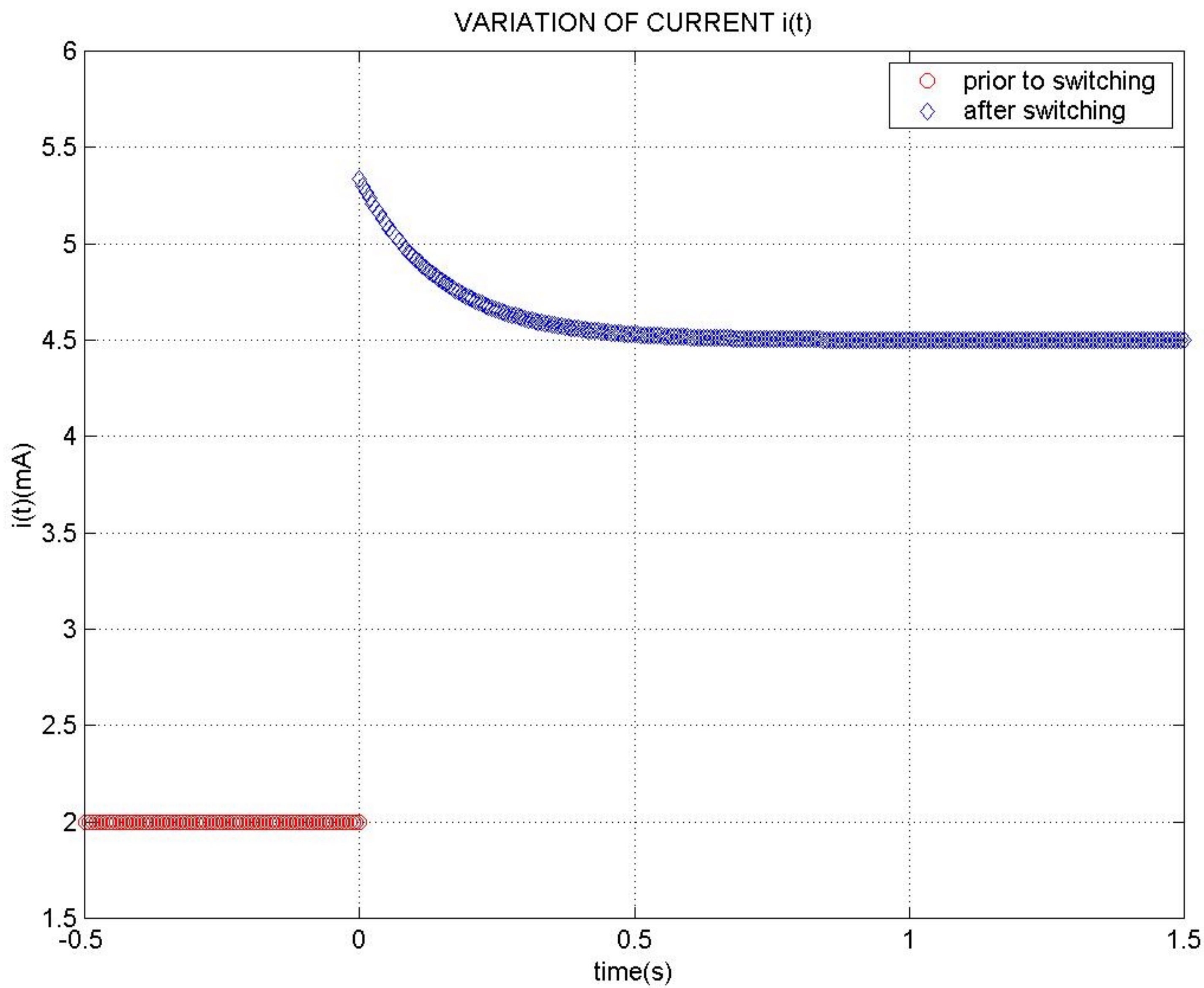
$$i(t) = \begin{cases} 2mA & t \le 0 \\ \frac{36}{8} + \frac{5}{6} e^{-\frac{t}{0.15}}, & t > 0 \end{cases}$$

**Command used to define linearly spaced arrays**

```
» help linspace
 LINSPACE Linearly spaced vector.
   LINSPACE(x1, x2) generates a row vector of 100 linearly
   equally spaced points between x1 and x2.
    LINSPACE(x1, x2, N) generates N points between x1 and x2.
    See also LOGSPACE, :.
```

**Script (m-file) with commands used. Prepared with the MATLAB Editor**

```
%example6p3.m
%commands used to display funtion i(t)
%this is an example of MATLAB script or M-file
%must be stored in a text file with extension ".m"
%the commands are executed when the name of the M-file is typed at the
%MATLAB prompt (without the extension)
tau=0.15; %define time constant
tini=-4*tau; %select left starting point
tend=10*tau; %define right end point
tminus=linspace(tini,0,100); %use 100 points for t<=0
tplus=linspace(0,tend, 250); % and 250 for t>=0
iminus=2*ones(size(tminus)); %define i for t<=0
iplus=36/8+5/6*exp(-tplus/tau); %define i for t>=0
plot(tminus,iminus,'ro',tplus,iplus,'bd'), grid; %basic plot command specifying
                                                  %color and marker
title('VARIATION OF CURRENT i(t)'), xlabel('time(s)'), ylabel('i(t)(mA)')
legend('prior to switching', 'after switching')
axis([-0.5,1.5,1.5,6]);%define scales for axis [xmin,xmax,ymin,ymax]
```


VARIATION OF CURRENT i(t)
prior to switching
after switching
i(t)(mA)
time(s)
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
-0.5
0
0.5
1
1.5

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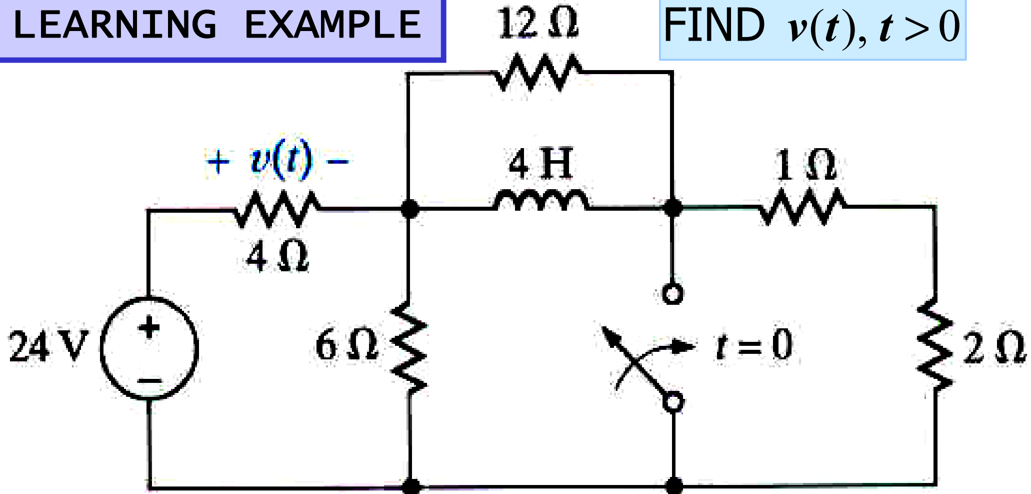
## LEARNING EXAMPLE

FIND $v(t), t>0$



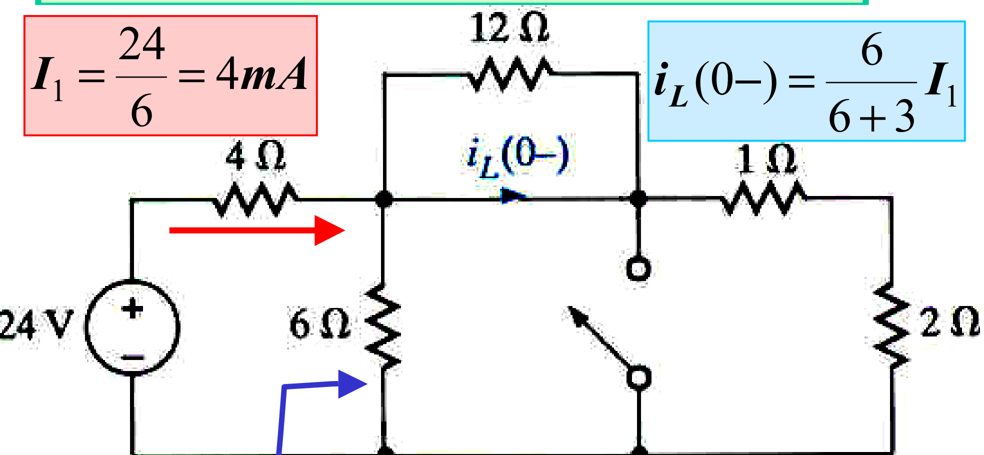

STEP 1: $v(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$

STEP 2: Initial inductor current

Use circuit in steady state prior to switching

$I_1 = \frac{24}{6} = 4mA$

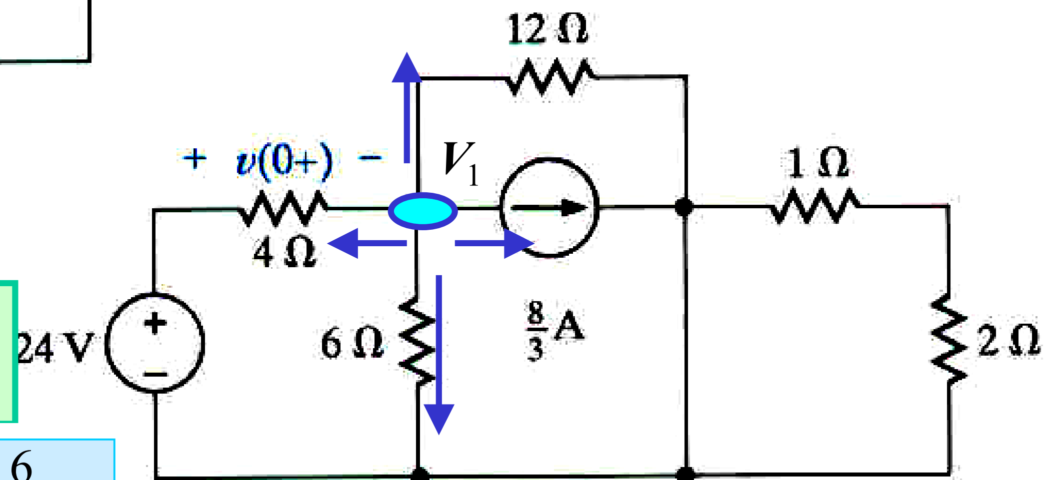
$i_L(0-) = \frac{6}{6+3} I_1$




$i_L(0-) = \frac{8}{3} mA$

STEP 3: Determine $v(0+)$

Use circuit at t=0+. Inductor is replaced by current source




$\frac{V_1 - 24}{4} + \frac{V_1}{6} + \frac{V_1}{12} + \frac{8}{3} = 0$

$V_1 = \frac{20}{3}[V]$

$v(0+) = 24[V] - V_1 = \frac{52}{3}[V]$

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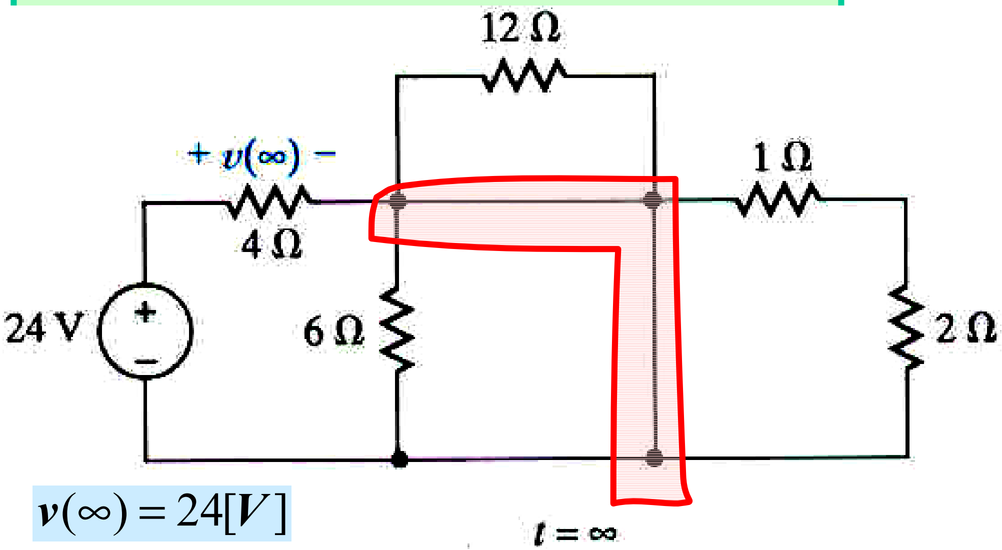
## STEP 4: DETERMINE $v(\infty)$

USE CIRCUIT IN STEADY STATE AFTER SWITCHING

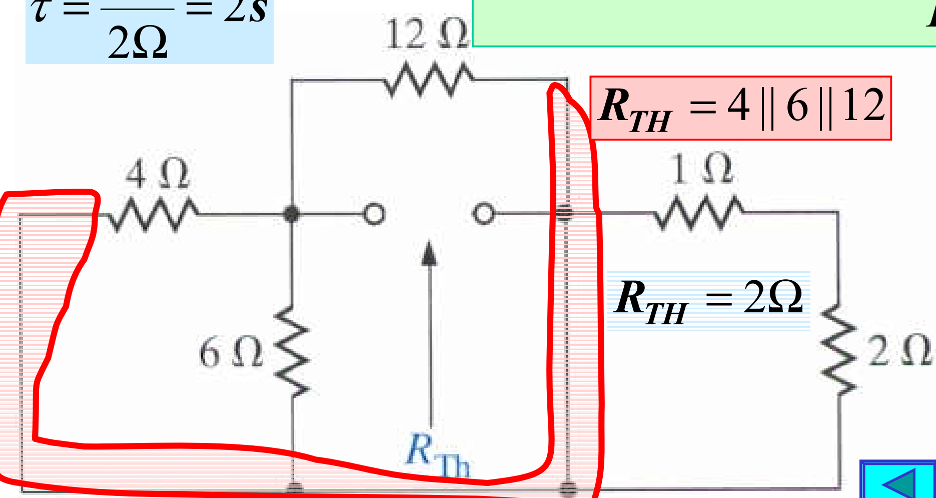



$v(\infty) = 24[V]$

## STEP 5: DETERMINE TIME CONSTANT

$$\tau = \frac{4H}{2\Omega} = 2s$$

Inductive Circuit : $\tau = \dfrac{L}{R_{TH}}$




$R_{TH} = 4 \| 6 \| 12$

$R_{TH} = 2\Omega$

## STEP 6: DETERMINE $K_1, K_2$

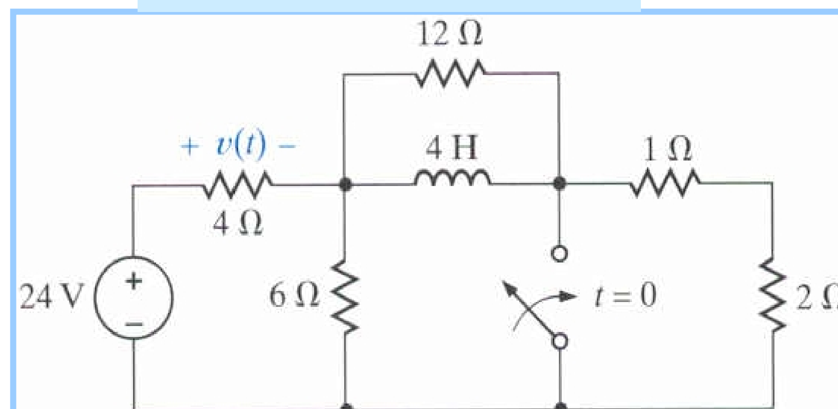
$K_1 = v(\infty) = 24[V]$ (step 4)

$v(0+) = \dfrac{52}{3} = K_1 + K_2$ (step 3)

$$K_1 = \frac{52}{3} - 24 = -\frac{20}{3}[V]$$

ANS: $v(t) = -\dfrac{20}{3} + 24e^{-\frac{t}{2}}, t > 0$

ORIGINAL CIRCUIT

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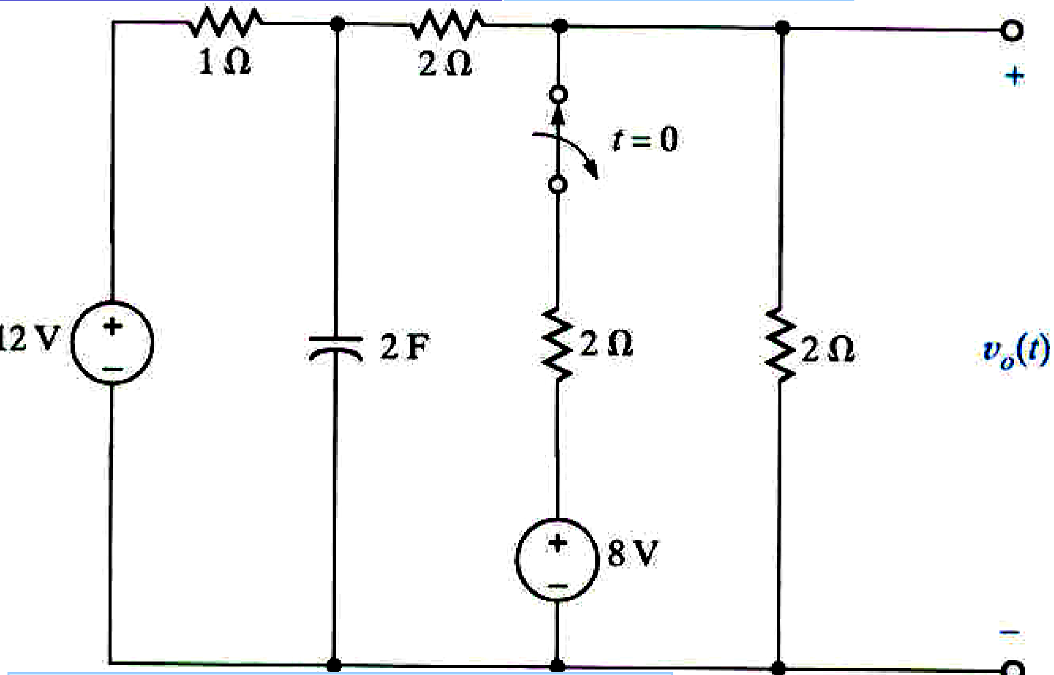
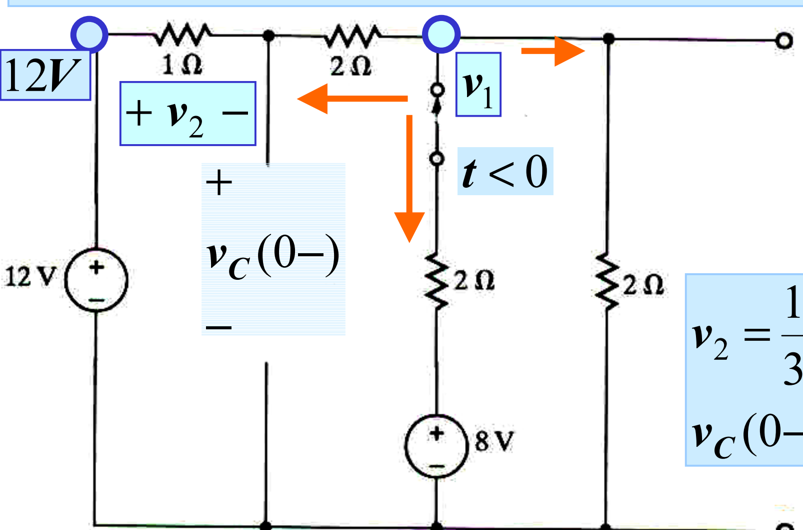
**LEARNING EXTENSION** FIND $v_O(t), t>0$




STEP 1: $v_o(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$

STEP 2 : INITIAL CAPACITOR VOLTAGE




$$v_2 = \frac{1}{3}(12 - v_1)$$

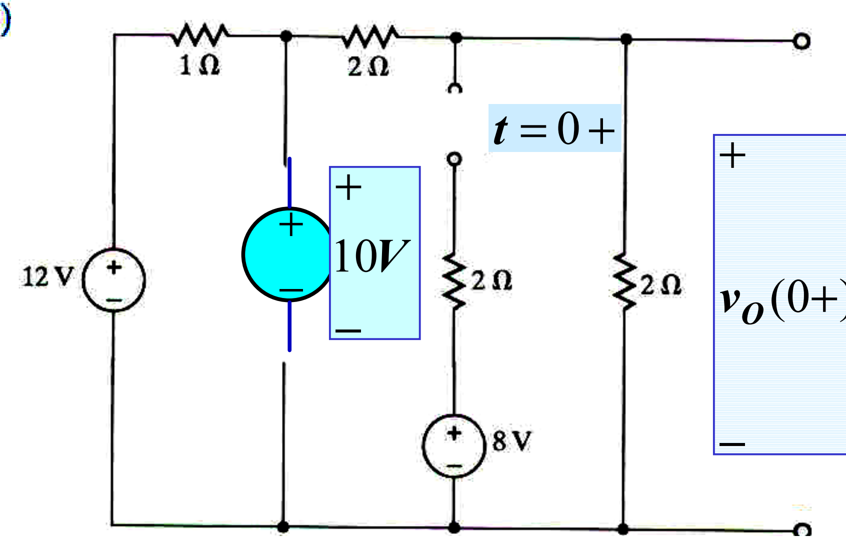
$$v_C(0-) = 12 - v_2$$

KCL @ $v_1$: $\frac{v_1 - 12}{3} + \frac{v_1 - 8}{2} + \frac{v_1}{2} = 0$ $*/6$

$$8v_1 - 48 = 0 \Rightarrow v_1 = 6[V]$$

$$v_2 = 2[V] \Rightarrow v_C(0-) = v_C(0+) = 10[V]$$

STEP 3: DETERMINE $v_O(0+)$




$$v_O(0+) = \frac{2}{2+2}(10) = 5[V]$$

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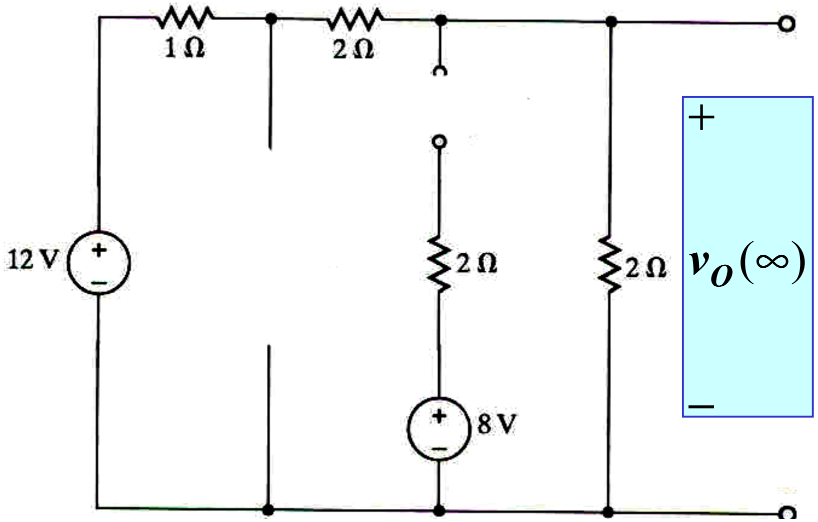
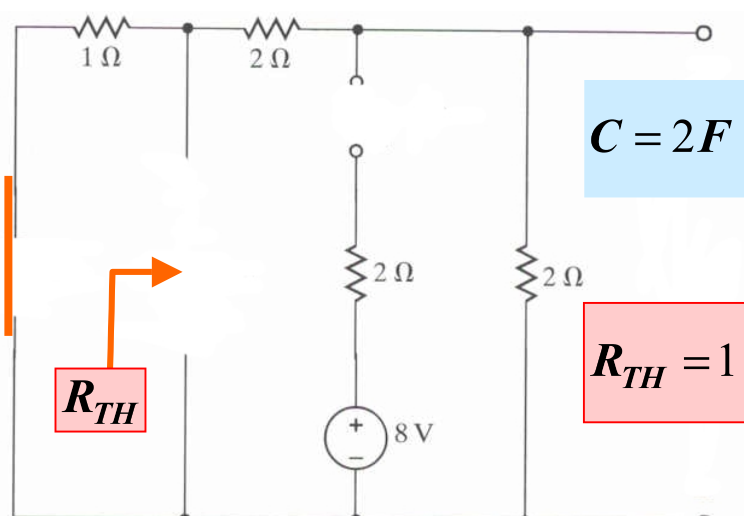
## STEP 4: DETERMINE $v_O(\infty)$




$$v_O(\infty) = \frac{2}{5}(12) = \frac{24}{5}[V]$$

## STEP 5: DETERMINE TIME CONSTANT

Capacitive Circuit: $\tau = R_{TH}C$




$$C = 2F \Rightarrow \tau = \frac{8}{5}s$$

$$R_{TH} = 1 \,\|\, 4 = \frac{4}{5}\Omega$$

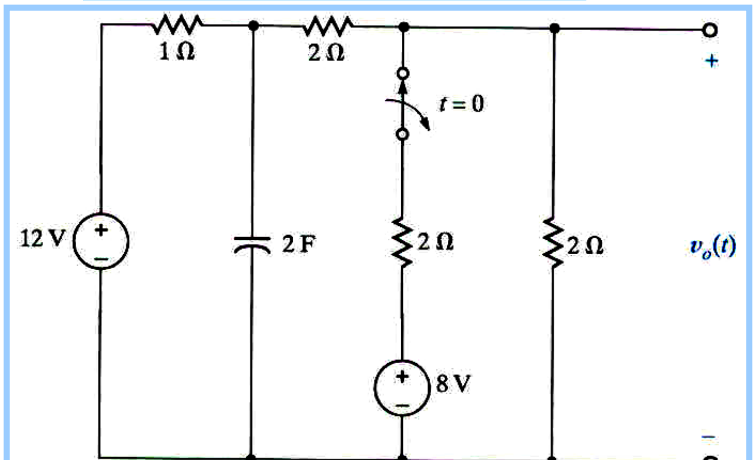
## STEP 6: DETERMINE $K_1, K_2$

$$K_1 = v_O(\infty) = \frac{24}{5}[V]$$

$$v_O(0+) = 5[V] = K_1 + K_2 \Rightarrow K_2 = \frac{1}{5}[V]$$

ANS: $v_O(t) = \frac{24}{5} + \frac{1}{5}e^{-\frac{t}{8/5}}[V];\ t > 0$

ORIGINAL CIRCUIT

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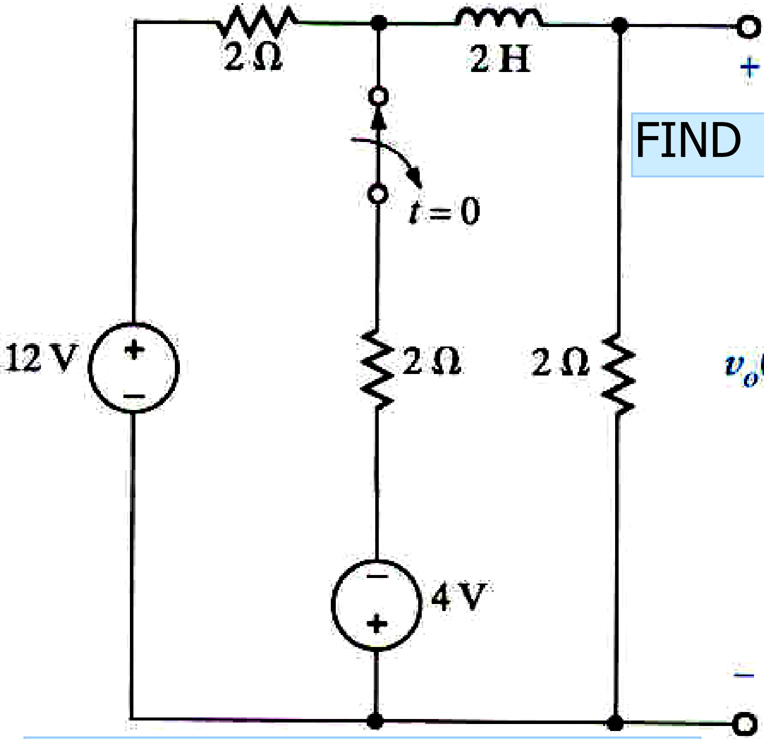
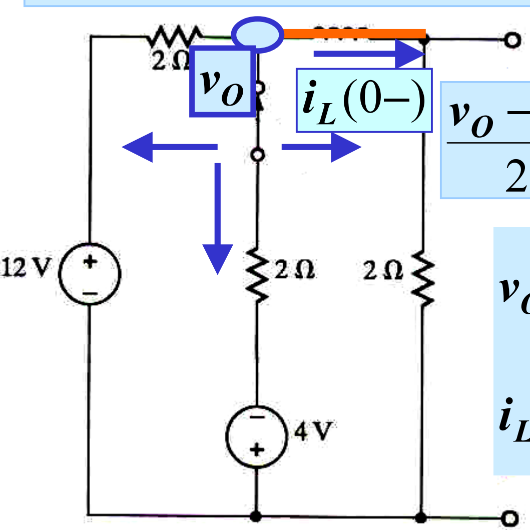



FIND $v_O(t), t > 0$

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: INITIAL INDUCTOR CURRENT

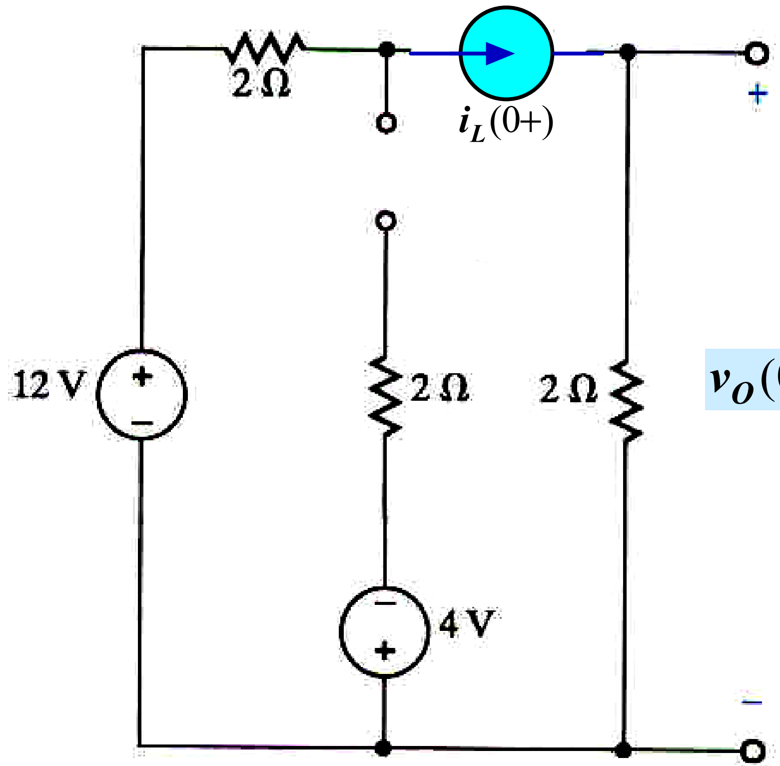



$$\frac{v_O - 12}{2} + \frac{v_O - (-4)}{2} + \frac{v_O}{2} = 0$$

$$v_O = \frac{8}{3}[V]$$

$$i_L(0-) = i_L(0+) = \frac{4}{3}[A]$$

STEP 3: DETERMINE $v_O(0+)$




$$v_O(0+) = 2 i_L(0+) = \frac{8}{3}[V]$$

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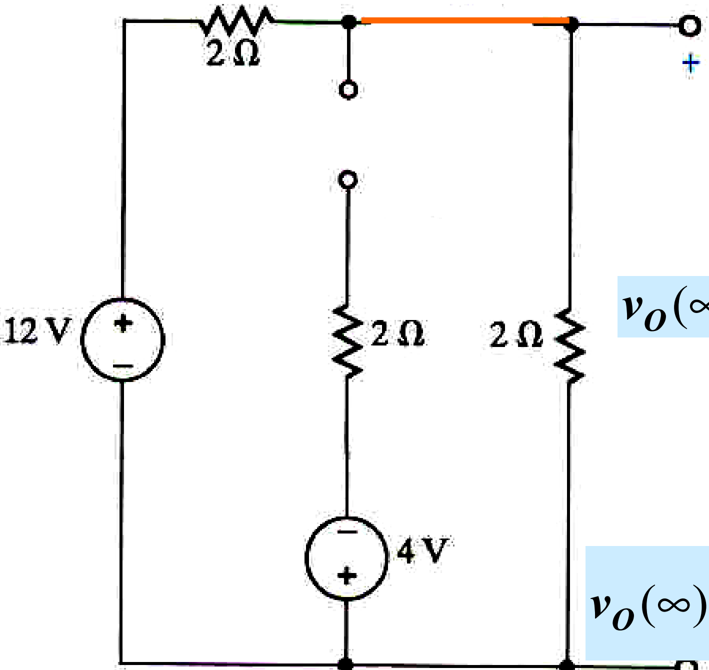
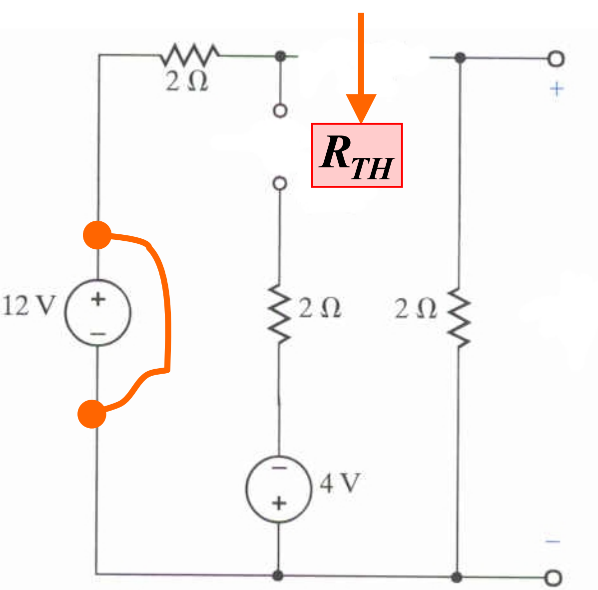
## STEP 4: DETERMINE $v_O(\infty)$




$v_O(\infty)$

$$v_O(\infty) = \frac{2}{2+2}(12) = 6[V]$$

## STEP 5: DETERMINE TIME CONSTANT




$R_{TH}$

Inductive Circuit

$$\tau = \frac{L}{R_{TH}}$$

$$R_{TH} = 4\Omega$$

$$\tau = \frac{2}{4} = 0.5s$$

## STEP 6: DETERMINE $K_1, K_2$

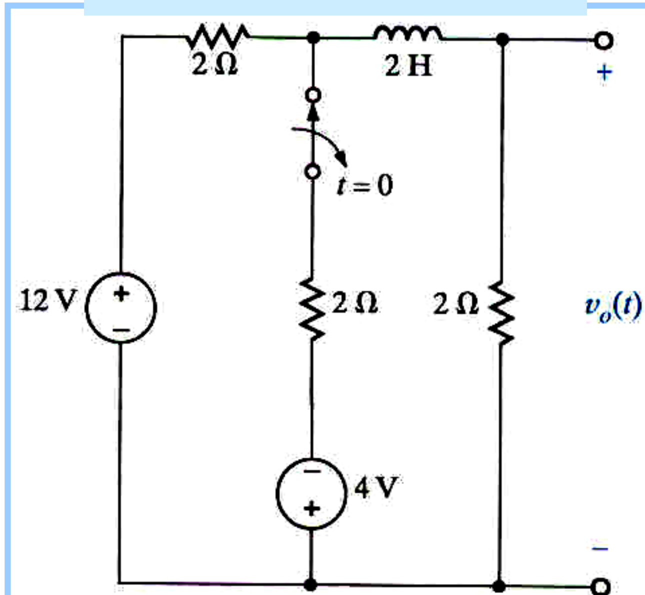
$K_1 = v_O(\infty) = 6[V]$ (step 4)

$v_O(+) = \frac{8}{3} = K_1 + K_2$ (step 3)

$$K_2 = \frac{8}{3} - 6 = -\frac{10}{3}[V]$$

ANS: $v_O(t) = 6 - \frac{10}{3}e^{-\frac{t}{0.5}}, t > 0$

ORIGINAL CIRCUIT

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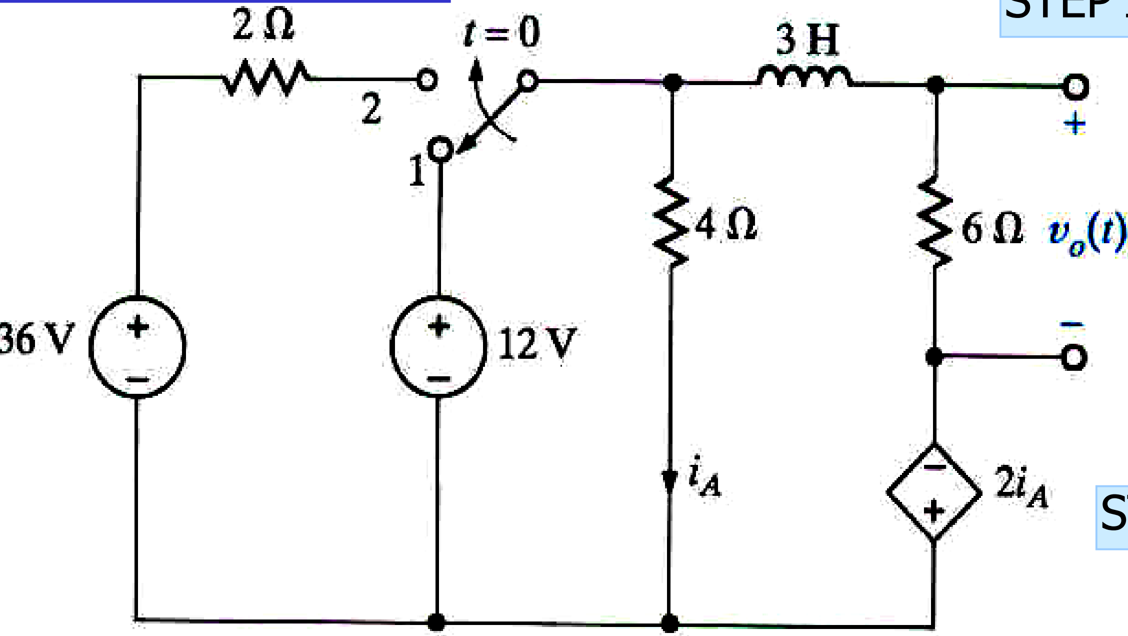
LEARNING EXAMPLE

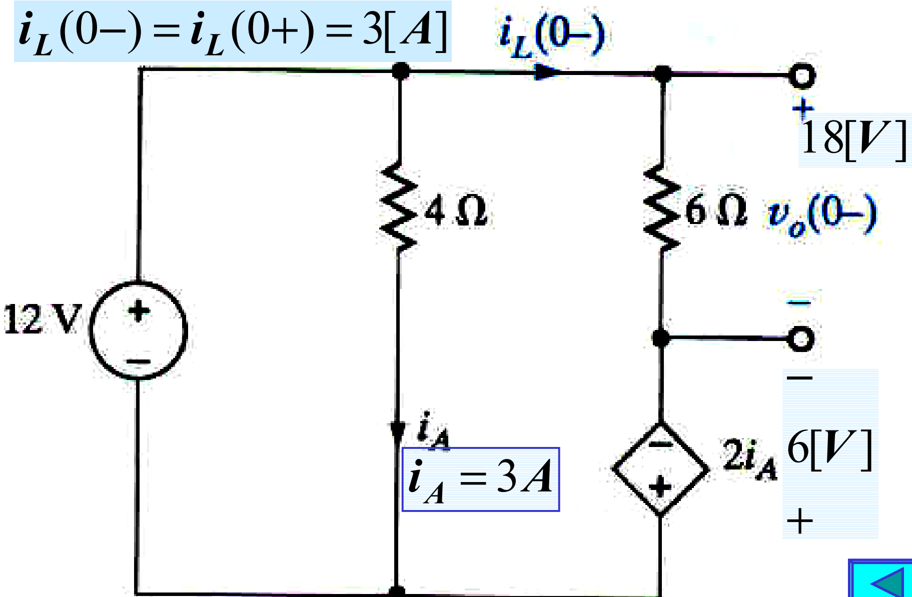
FIND $v_O(t), t > 0$



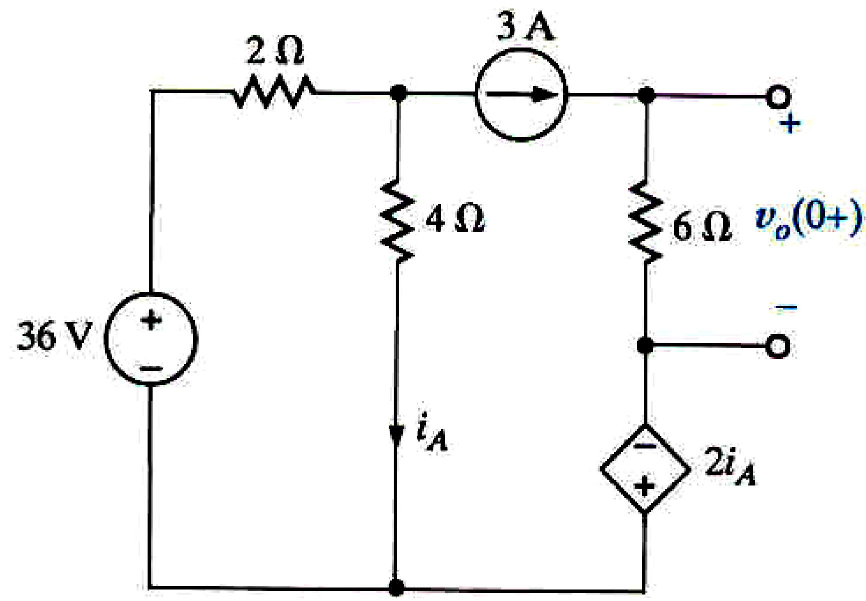

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: DETERMINE $i_L(0+)$

$i_L(0-) = i_L(0+) = 3[A]$




STEP 3: DETERMINE $v_O(0+)$




$v_O(0+) = 18[V]$

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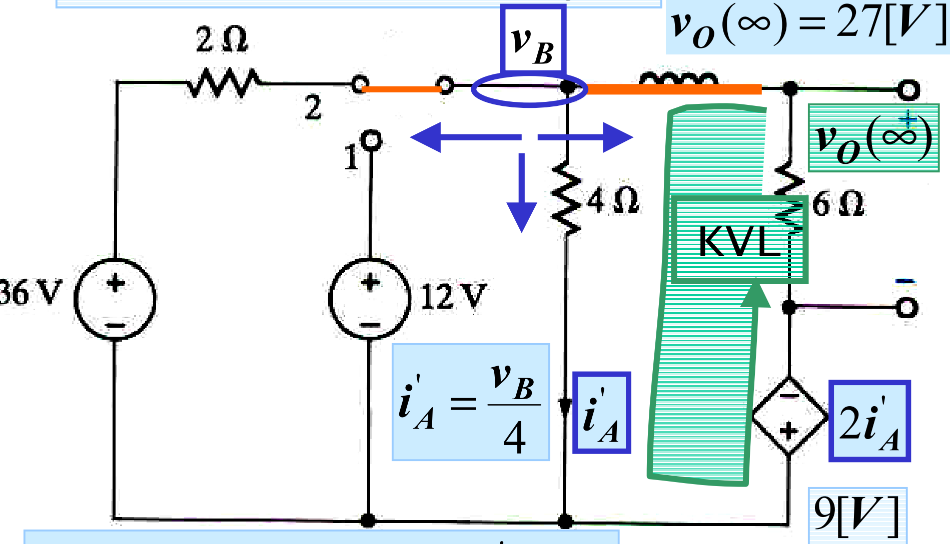
# STEP 4: DETERMINE $v_O(\infty)$

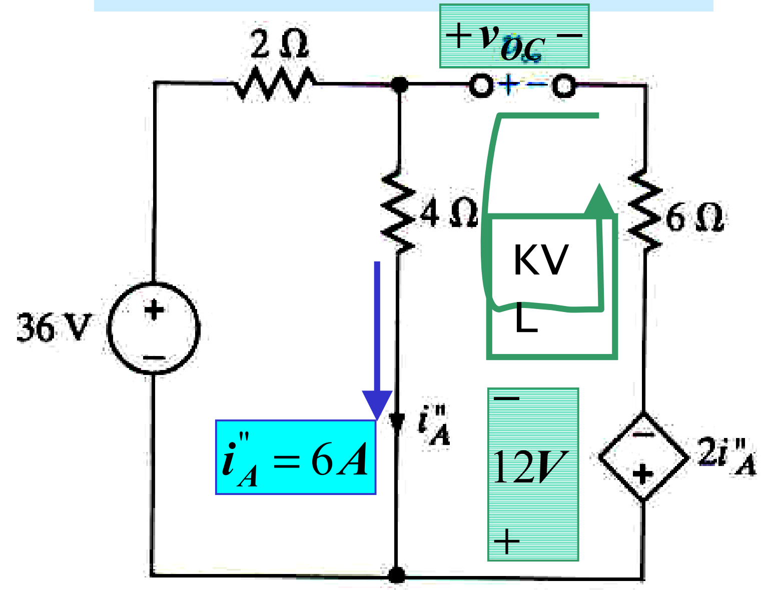
$v_O(\infty) = 27[V]$




$$\frac{v_B - 36}{2} + \frac{v_B}{4} + \frac{v_B - (-2i_A')}{6} = 0 \; */12$$

$$11v_B + 4i_A' = 36 \times 6 \qquad v_B = 18[V],\; i_A' = 4.5[A]$$

# OPEN CIRCUIT VOLTAGE




$$v_{OC} = 24 + 12 = 36[V]$$

# STEP 5: DETERMINE TIME CONSTANT

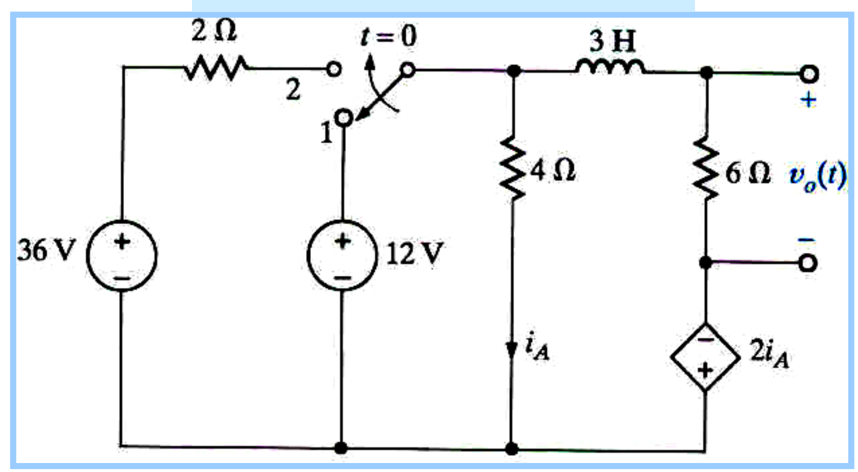
inductive circuit

$$\tau = \frac{L}{R_{TH}}$$

Circuit with dependent sources

$$R_{TH} = \frac{v_{OC}}{i_{SC}}$$

# ORIGINAL CIRCUIT

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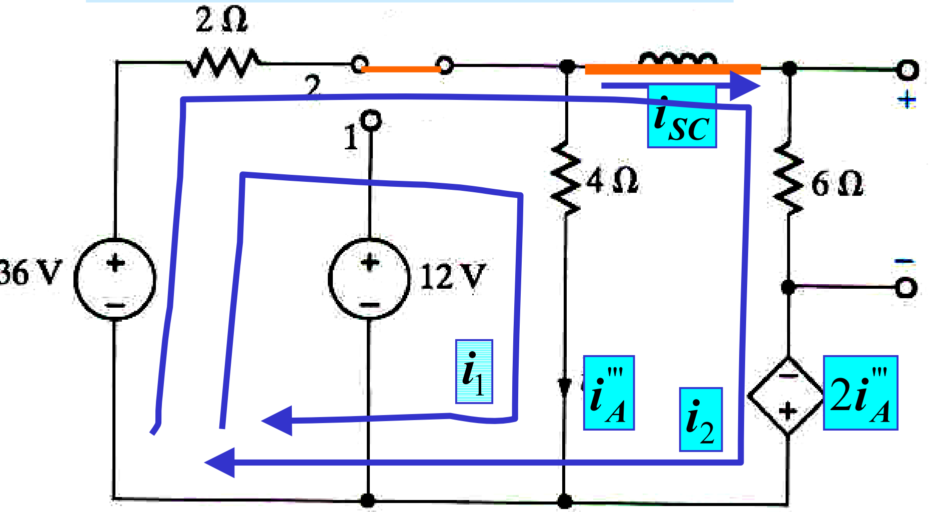
## SHORT CIRCUIT CURRENT




NOTE: FOR THE INDUCTIVE CASE THE CIRCUIT USED TO COMPUTE THE SHORT CIRCUIT CURRENT IS THE SAME USE TO DETERMINE $v_O(\infty)$

$$36 = 2(i_1 + i_2) + 4i_1$$

$$36 = 2(i_1 + i_2) + 6i_2 - 2i_A'''$$

$$i_A''' = i_1$$

$$i_{SC} = \frac{36}{8}[A]$$

$$\left.\begin{array}{l} v_{OC} = 36[V] \\ i_{SC} = 36/8[A] \end{array}\right\} \Rightarrow R_{TH} = 8\Omega$$

$$L = 3H \Rightarrow \tau = \frac{3}{8}s$$

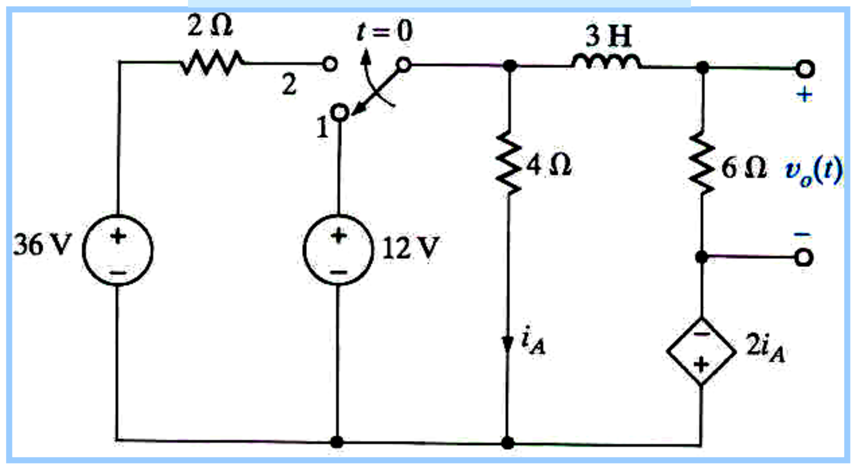
STEP 6: DETERMINE $K_1, K_2$

$v_O(\infty) = 27 = K_1$ (step 4)

$v_O(0+) = 18 = K_1 + K_2 \Rightarrow K_2 = -9[V]$ (step 3)

ANS: $v_O(t) = 27 - 9e^{-\frac{t}{3/8}}, \; t > 0$

## ORIGINAL CIRCUIT

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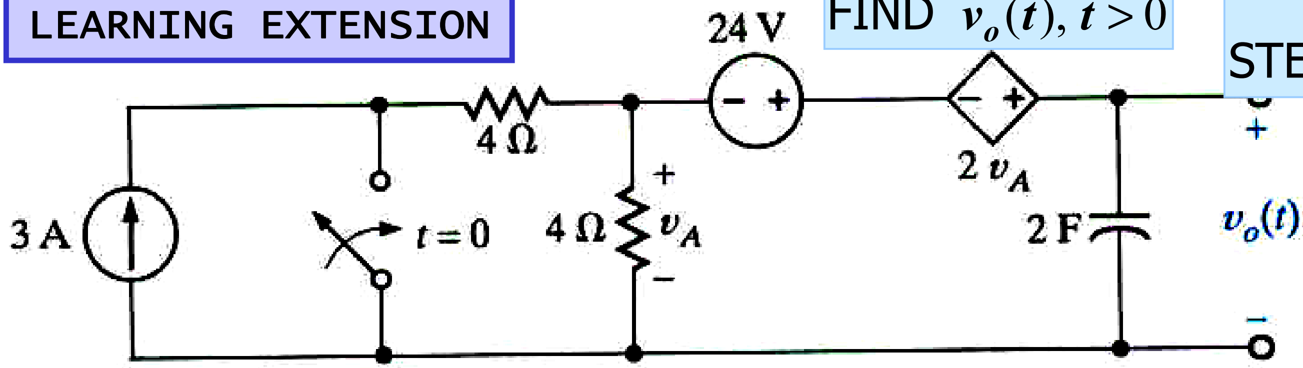
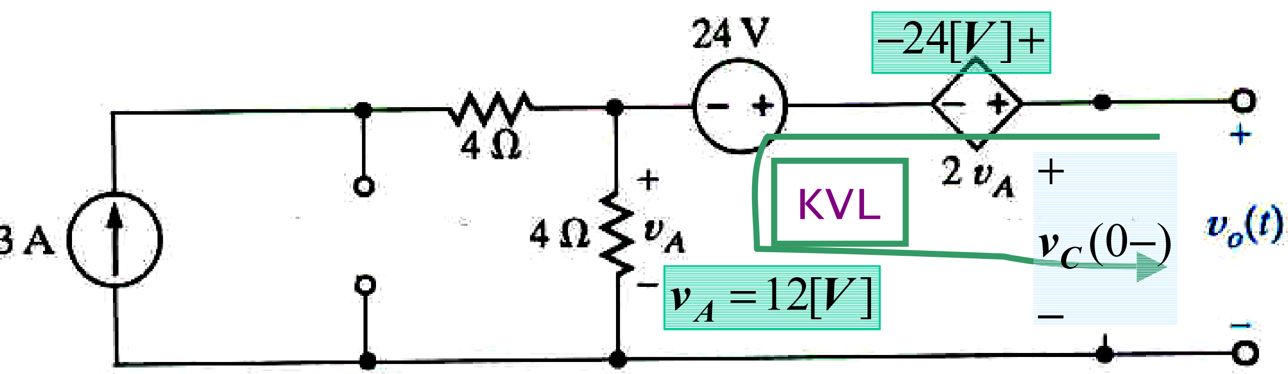
## LEARNING EXTENSION

FIND $v_o(t),\ t>0$

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}},\ t>0$




STEP 2: DETERMINE CAPACITOR VOLTAGE AT $t = 0+$



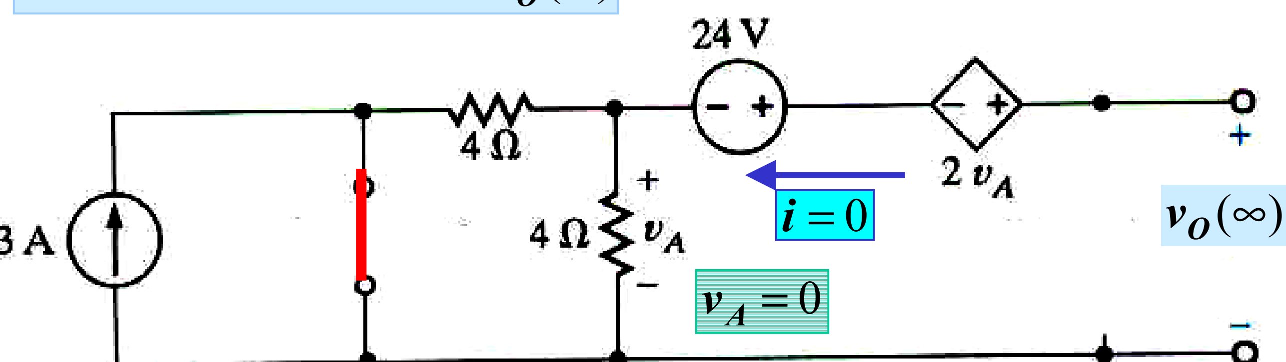

$v_C(0-) = 24 + 24 + 12 = 60[V]$

$v_C(0+) = v_C(0-)$

STEP 3: DETERMINE $v_O(0+)$

$v_O = v_C \Rightarrow v_O(0+) = 60[V]$

STEP 4: DETERMINE $v_O(\infty)$




$v_O(\infty) = 24[V]$

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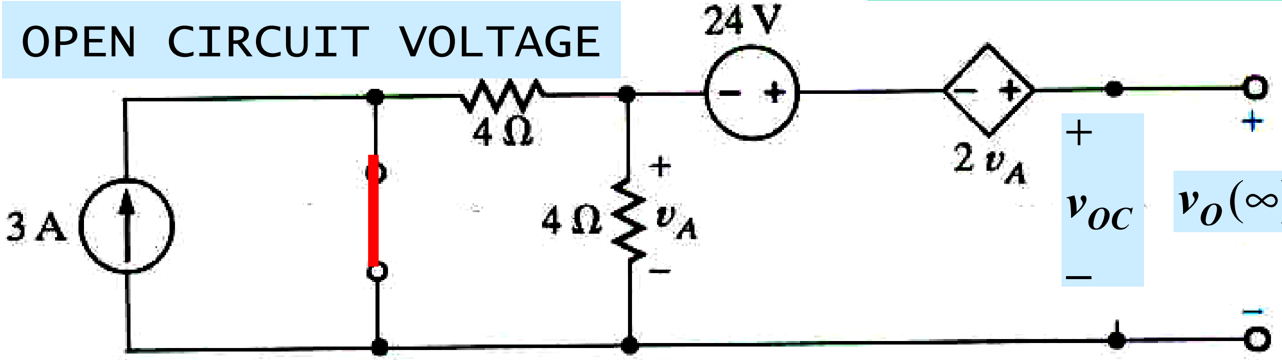
STEP 5: DETERMINE TIME CONSTANT

capacitive circuit $\Rightarrow \tau = R_{TH}C$

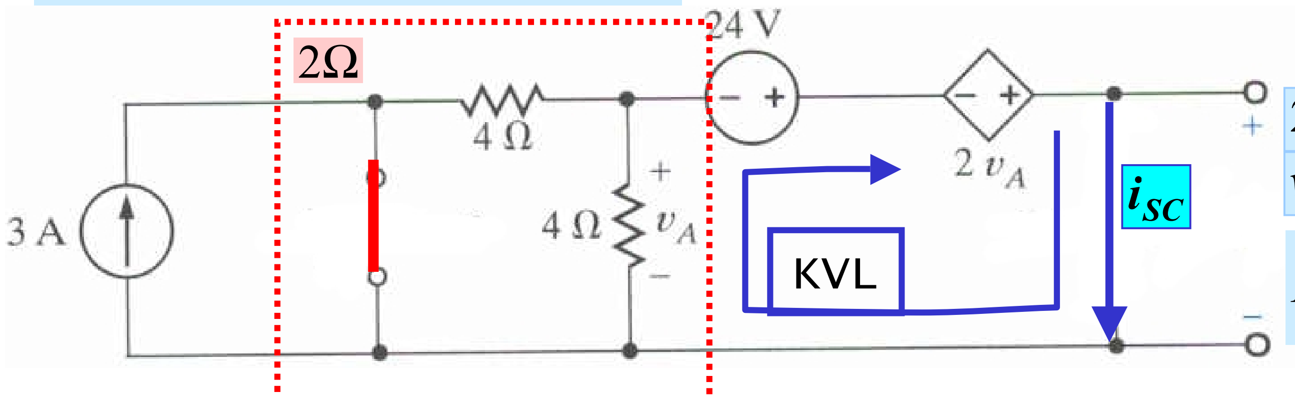
$$R_{TH} = \frac{v_{OC}}{i_{SC}}$$

OPEN CIRCUIT VOLTAGE




$v_{OC} = v_O(\infty) = 24[V]$

SHORT CIRCUIT CURRENT




$2i_{SC} - 24 - 2v_A = 0$

$v_A = -2i_{SC}$

$i_{SC} = 4[A]$

$$R_{TH} = \frac{24}{4} = 6\Omega$$

$\tau = 6\Omega \times 2F = 12s$

STEP 6: DETERMINE $K_1, K_2$

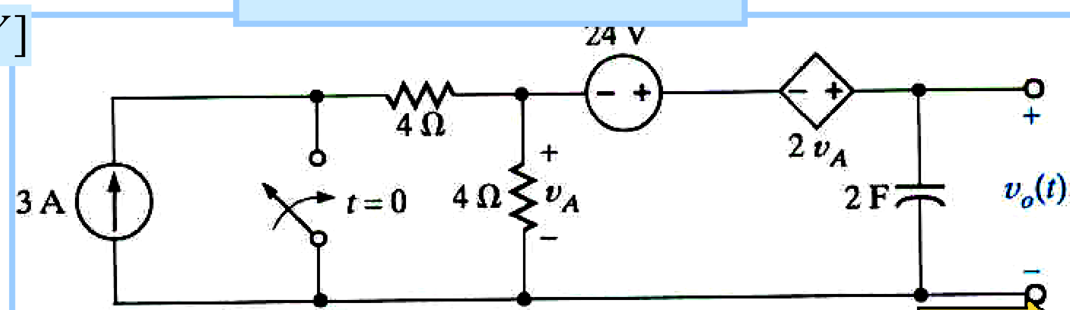
$K_1 = v_O(\infty) = 24$ (step 4)

$v_O(0+) = 60 = K_1 + K_2$ (step 3)

$K_2 = 36[V]$

ANS: $v_O(t) = 24 + 36e^{-\frac{t}{12}}, t > 0$

ORIGINAL CIRCUIT

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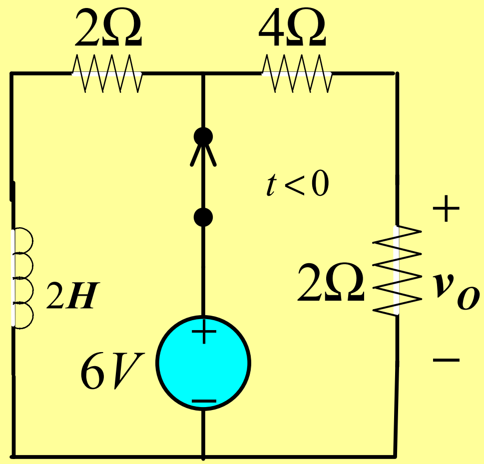
# Inductor example

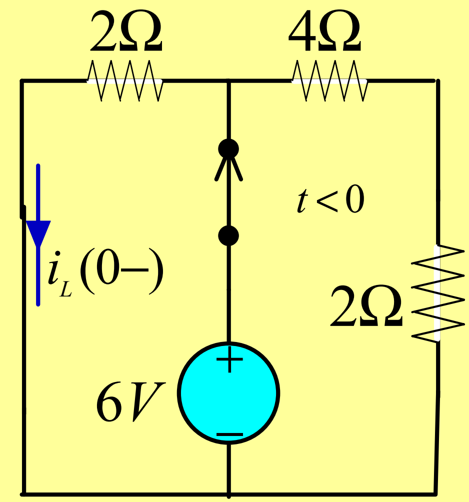
FIND $v_O(t), t > 0$




## STEP 1: Form of the solution

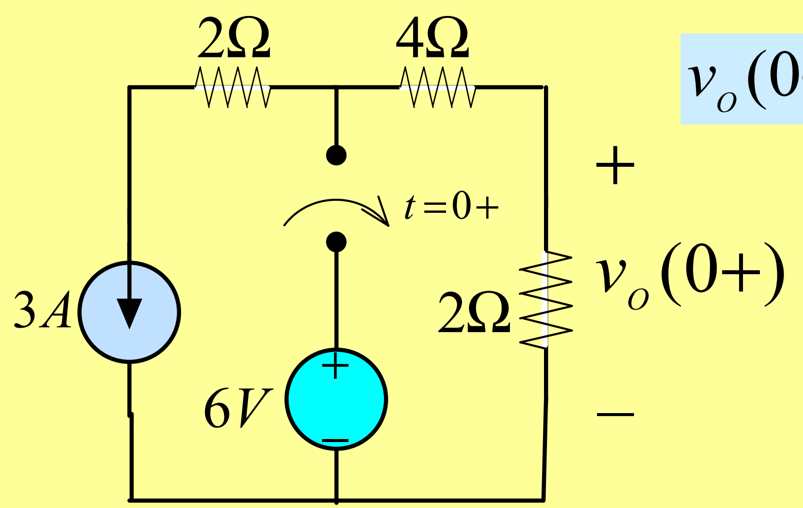
$$v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$$

## STEP 2: Initial inductor current




$$i_L(0-) = 3A$$

## STEP 3: Determine output at 0+ (inductor current is constant)




$$v_O(0+) = -6V$$

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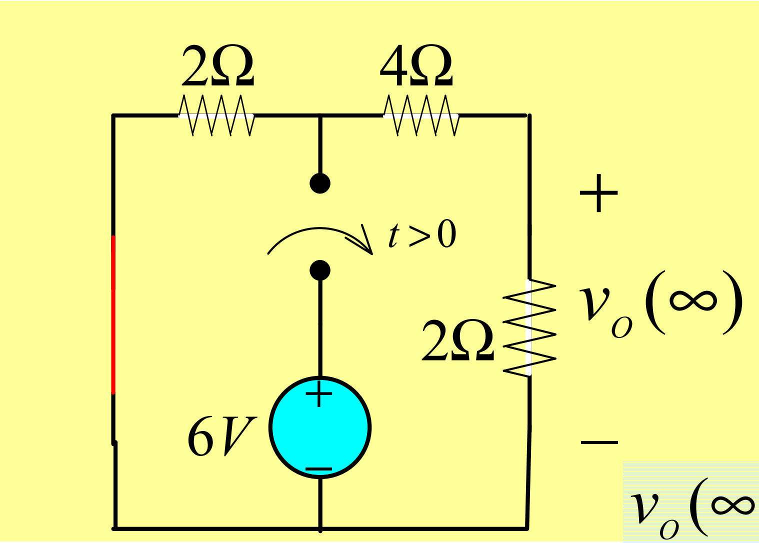
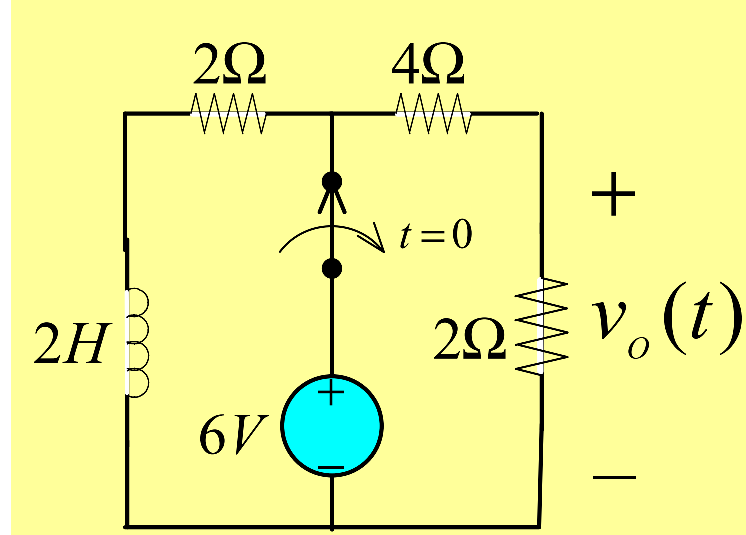
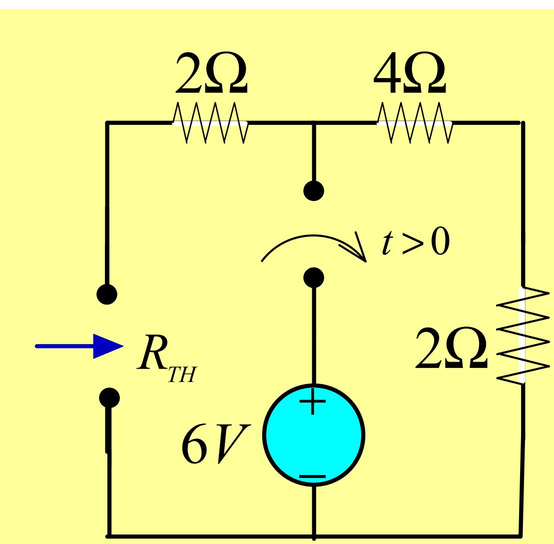
STEP 4: Find output in steady state after the switching




$$v_O(\infty) = 0$$




STEP 5: Find time constant after switch




$$\tau = \frac{L}{R_{TH}}$$

$$R_{TH} = 8\Omega$$

$$\tau = 0.25\,s$$

STEP 6: Find the solution

$$K_1 + K_2 = v_O(0+) = -6V$$

$$K_1 = v_O(\infty) = 0$$

$$v_O(t) = -6e^{-\frac{t}{0.25}};\ t > 0$$

$$\boldsymbol{v_O(t) = -6e^{-4t};\ t > 0}$$

Pulse Response

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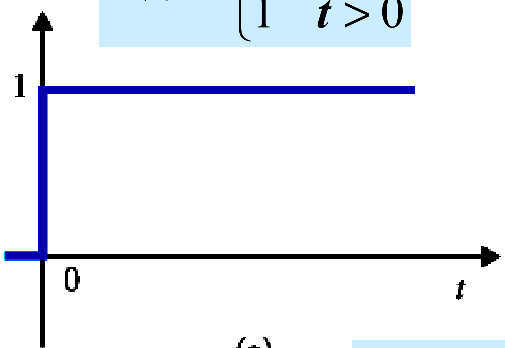
Step by Step

STOP

# PULSE RESPONSE

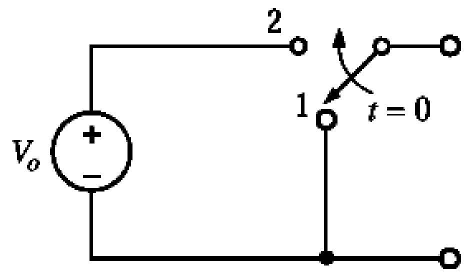
**WE STUDY THE RESPONSE OF CIRCUITS TO A SPECIAL CLASS OF *SINGULARITY FUNCTIONS***

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

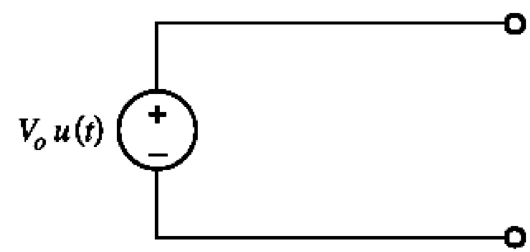



(a)

VOLTAGE STEP

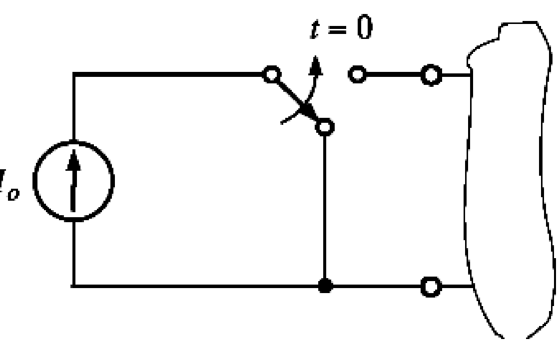


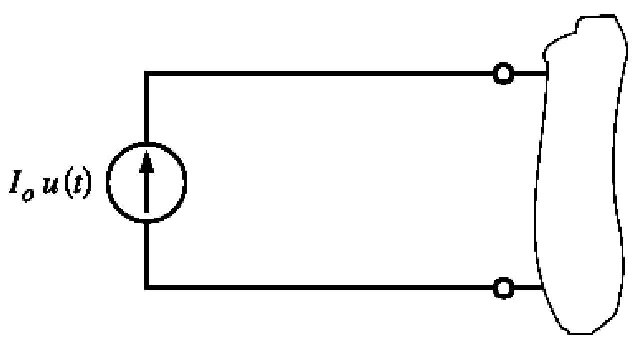

(b)



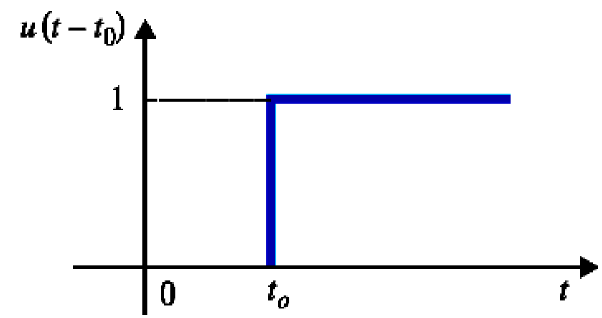

(c)

CURRENT STEP




TIME SHIFTED STEP

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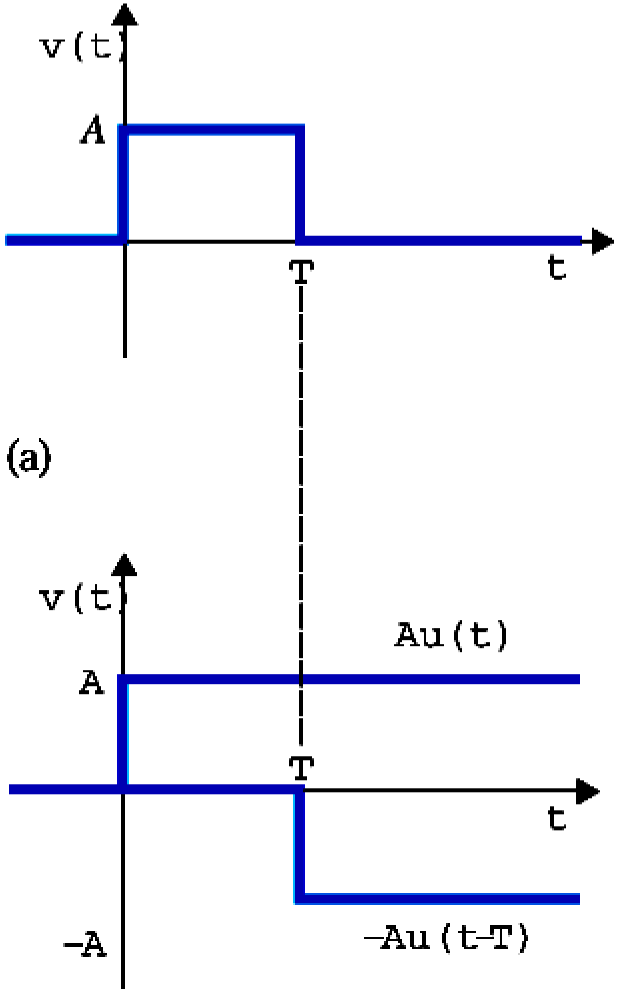
## PULSE SIGNAL

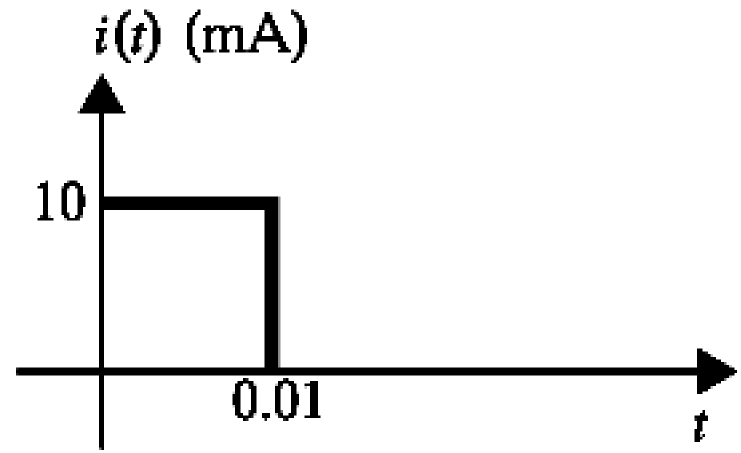



(b)

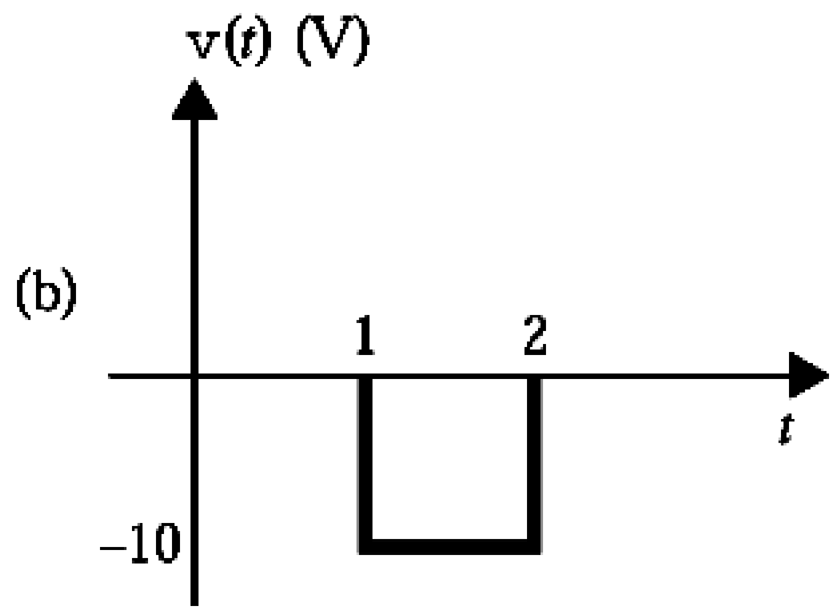
## PULSE AS SUM OF STEPS

## LEARN BY DOING

$$i(t) = 10[u(t) - u(t - 0.01)](mA)$$




$$v(t) = -10[u(t-1) - u(t-2)](V)$$

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# NONZERO INITIAL TIME AND REPEATED SWITCHING

$$\tau \frac{dx}{dt} + x = f_{TH}; \;\; x(t_0+) = x_0$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{1}{\tau} \int_{t_0}^{t} e^{-\frac{t-x}{\tau}} f_{TH}(x)dx$$

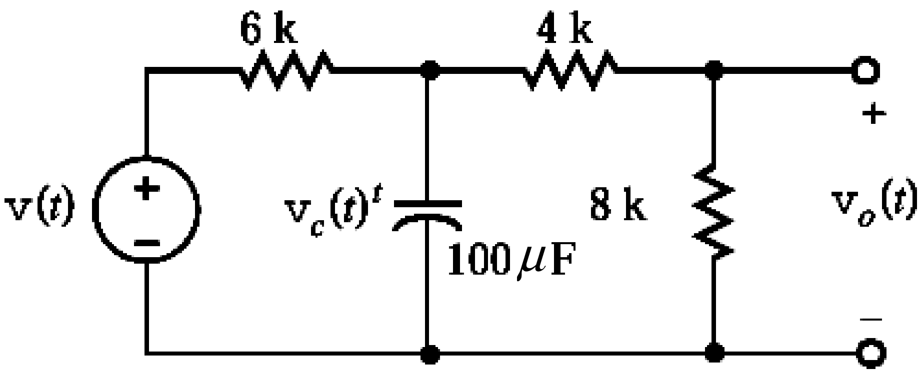
$$x(t) = K_1 + K_2 e^{-\frac{t-t_0}{\tau}}; \; t \geq t_0$$

RESPONSE FOR CONSTANT SOURCES

This expression will hold on ANY interval where the sources are constant. The values of the constants may be different and must be evaluated for each interval

The values at the end of one interval will serve as initial conditions for the next interval

LEARNING EXAMPLE FIND THE OUTPUT VOLTAGE $v_O(t);\ t>0$

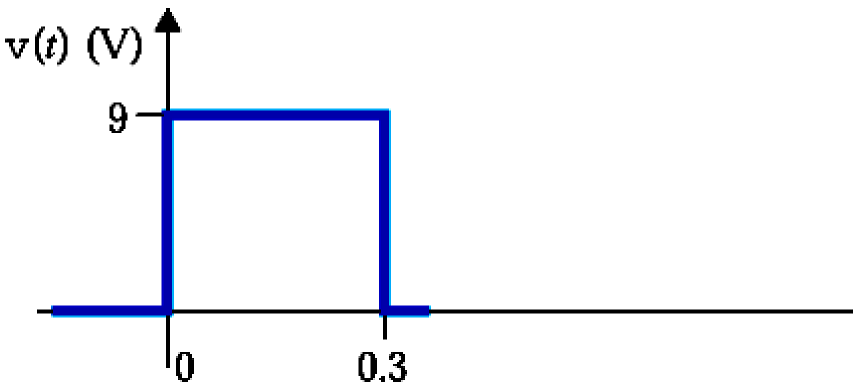


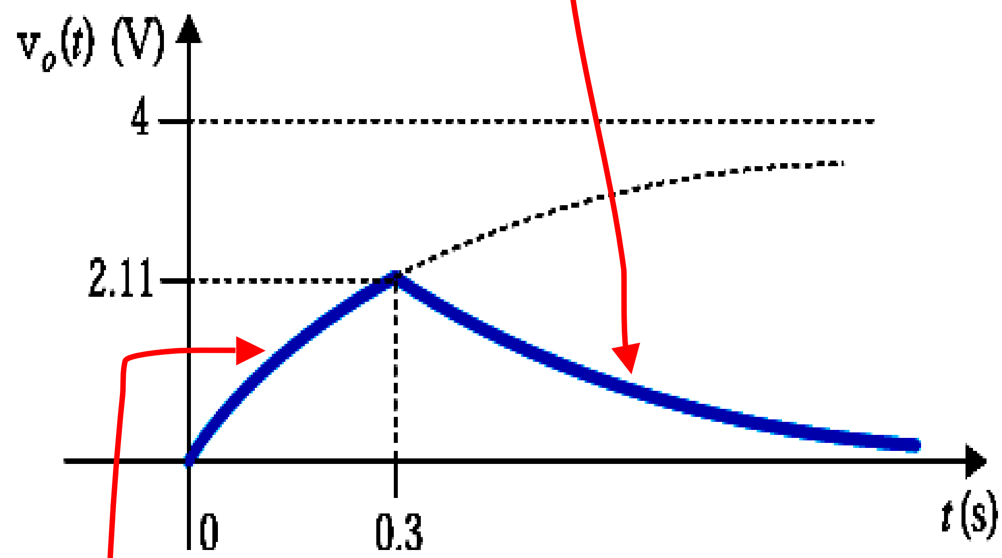

$t>0.3 \Rightarrow v(t)=0$ $t_o = 0.3$ $v_o(0.3+) = 4(1-e^{-\frac{0.3}{0.4}})$

$v_O(t) = K_1'' + K_2'' e^{-\frac{(t-0.3)}{\tau'}}$ $\tau' = 0.4$

$v_o(\infty) = 0 \Rightarrow K_1'' = 0$ $K_2'' = v_o(0.3+) = 2.11(V)$

$v_o(t) = 2.11 e^{-\frac{t-0.3}{0.4}};\ t>0.3$




$t<0 \Rightarrow v(t)=0 \Rightarrow v_O(t)=0$ $v_O(0+)=0$

$t>0 \Rightarrow v(t) = 9V$

$v_o(t) = K_1' + K_2' e^{-\frac{t}{\tau}}$

$\tau = R_{TH}C = (6k \| 12k) \times 100\mu F = 0.4s$

$v_o(\infty) = \frac{8}{10+8}(9) = K_1'$ $v_o(0+) = K_1' + K_2' = 0$

$v_o(t) = 4\left(1-e^{-\frac{t}{0.4}}\right)$

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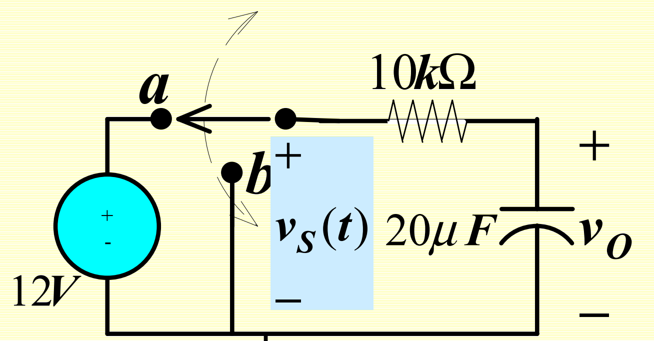
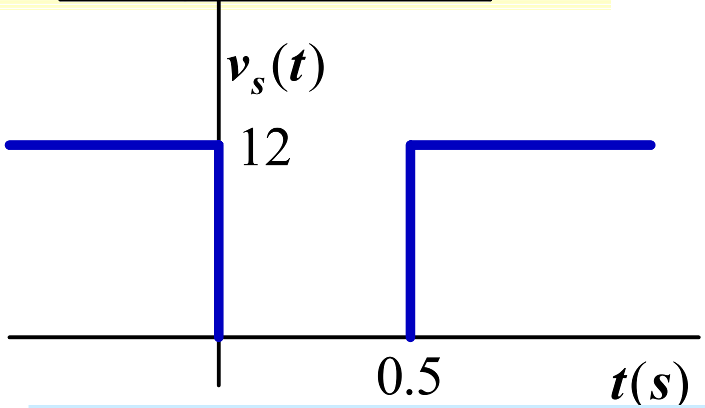
EXAMPLE

THE SWITCH IS INITIALLY AT a. AT TIME t=0 IT MOVES TO b AND AT t=0.5 IT MOVES BACK TO a. FIND $v_O(t),\, t>0$




Piecewise constant source

ON EACH INTERVAL WHERE THE SOURCE IS CONSTANT THE OUTPUT IS OF THE FORM

$$v_O(t) = K_1 + K_2 e^{-\frac{t-t_O}{\tau}}$$

FOR $0 < t < 0.5$ (switch at $b$) $t_o = 0$

$$v_O(t) = K_1' + K_2' e^{-\frac{t}{\tau}}$$

$$v(0+) = 12[V] = K_1' + K_2'$$

$$v_O(\infty) = 0 = K_1'$$

$$\tau = (10k\Omega)(20\mu F) = 0.2s$$

$$v_O(t) = 12e^{-\frac{t}{0.2}}, 0 < t < 0.5$$

FOR $t > 0.5$ (switch at $a$) $t_o = 0.5$

$$v_O(0.5+) = v_O(0.5-) = 12e^{-\frac{0.5}{0.2}} = 0.985$$

$$v_O(t) = K_1'' + K_2'' e^{-\frac{(t-0.5)}{\tau'}}$$

$$v_O(0.5+) = 0.985 = K_1'' + K_2''$$

$$v_O(\infty) = 12 = K_1''$$

$$K_2'' = 0.985 - 12 = -11.015$$

$$v_O(t) = 12 - 11.015e^{-\frac{t-0.5}{0.2}}, t > 0.5$$

The constants are determined in the usual manner

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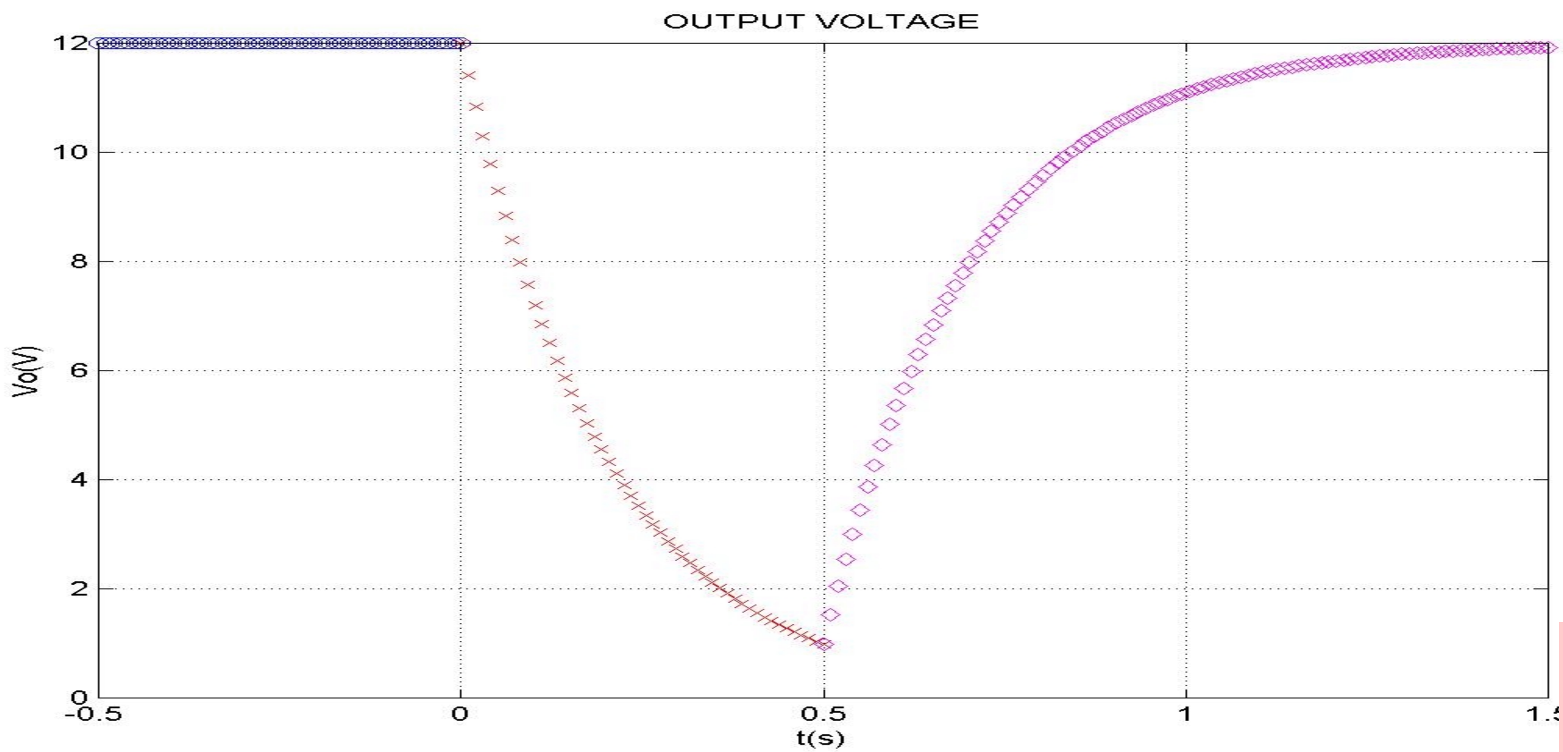
# USING MATLAB TO DISPLAY OUTPUT VOLTAGE

```
%pulse1.m
% displays the response to a pulse response
tmin=linspace(-0.5,0,50); %negative time segment
t1=linspace(0,0.5,50); %first segment
t2=linspace(0.5, 1.5,100); %second segment
vomin=12*ones(size(tmin));
vo1=12*exp(-t1/0.2); %after first switching
vo2=12-11.015*exp(-(t2-0.5)/0.2); %after second switching
plot(tmin,vomin,'bo',t1,vo1,'rx',t2,vo2,'md'),grid
title('OUTPUT VOLTAGE'), xlabel('t(s)'),ylabel('Vo(V)')
```




First Order

STOP