13.4 If $f(t) = e^{-at}\sin\omega t$, show that $\mathbf{F}(s) = \dfrac{\omega}{(s+a)^2+\omega^2}$.

SOLUTION:

$$F(s) = \int_0^\infty e^{-at} e^{-st} \sin\omega t\, dt = \int_0^\infty e^{-(s+a)t} \sin\omega t\, dt$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$F(s) = \int_0^\infty e^{-(s+a)t} \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\, dt = \int_0^\infty \frac{e^{(j\omega - s - a)t} - e^{(-j\omega - s - a)t}}{2j}\, dt$$

$$F(s) = \left. \frac{e^{(j\omega - s - a)t}}{(j\omega - s - a)2j} + \frac{e^{(-j\omega - s - a)t}}{(j\omega + s + a)2j} \right|_0^\infty = \frac{-1}{(j\omega - s - a)2j} - \frac{1}{(j\omega + s + a)j2}$$

$$F(s) = \left[\frac{1}{\omega + j(s+a)} + \frac{1}{\omega - j(s+a)}\right]\frac{1}{2} = \frac{\omega}{(\omega + j(s+a))(\omega - j(s+a))}$$

$$\boxed{F(s) = \frac{\omega}{(s+a)^2 + \omega^2}}$$