Discrete-Time Signals: Time-Domain Representation

- Signals represented as sequences of numbers, called **samples**
- Sample value of a typical signal or sequence denoted as $x[n]$ with n being an integer in the range $-\infty \leq n \leq \infty$
- $x[n]$ defined only for integer values of n and undefined for noninteger values of n
- Discrete-time signal represented by $\{x[n]\}$

Discrete-Time Signals: Time-Domain Representation

- Discrete-time signal may also be written as a sequence of numbers inside braces:

$$\{x[n]\} = \{\ldots, -0.2, \underset{\uparrow}{2.2}, 1.1, 0.2, -3.7, 2.9, \ldots\}$$

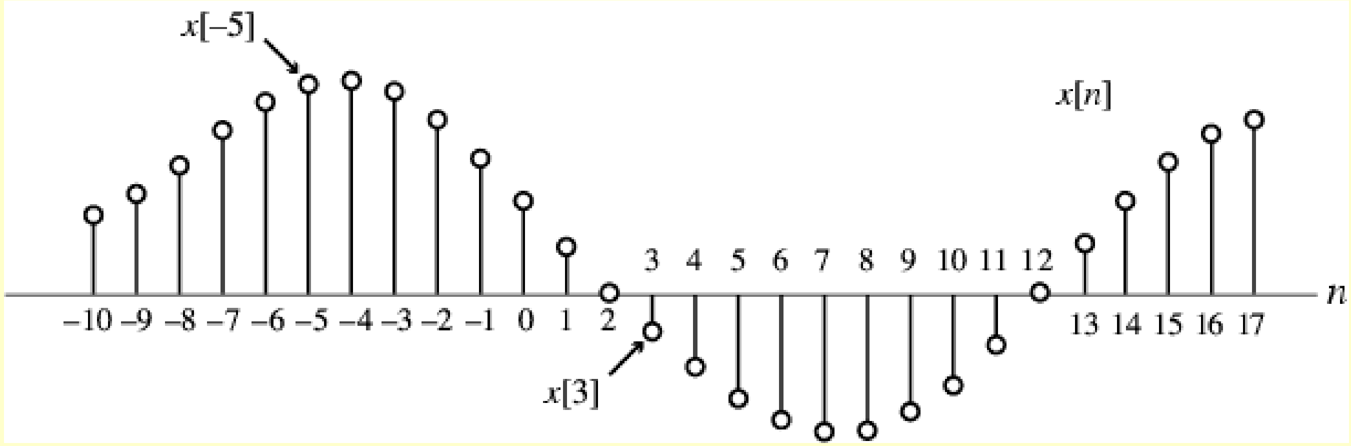
- In the above, $x[-1] = -0.2$, $x[0] = 2.2$, $x[1] = 1.1$,

etc.

- The arrow is placed under the sample at time index $n = 0$

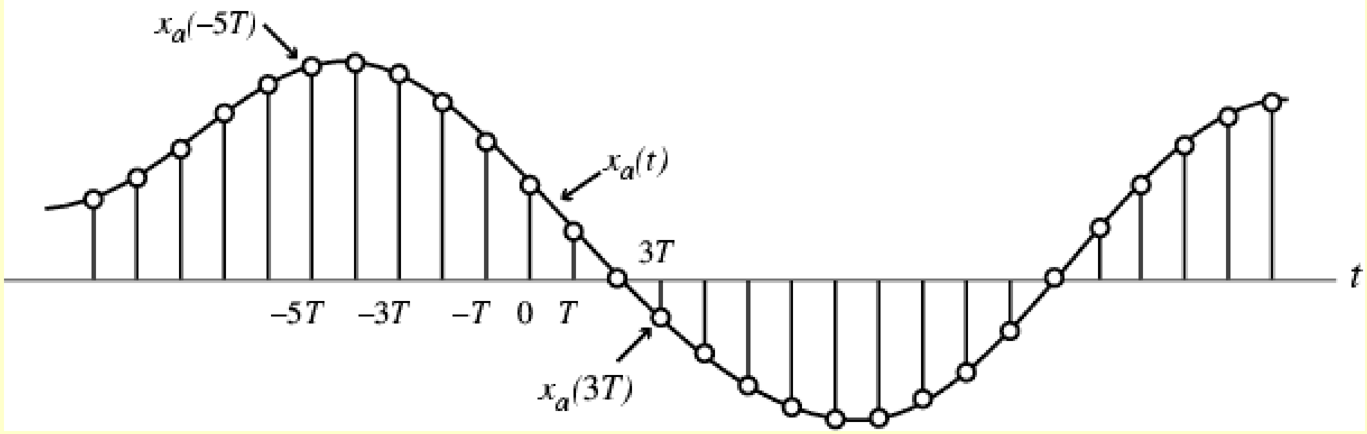
Discrete-Time Signals: Time-Domain Representation

- Graphical representation of a discrete-time signal with real-valued samples is as shown below:



Discrete-Time Signals: Time-Domain Representation

- In some applications, a discrete-time sequence $\{x[n]\}$ may be generated by periodically sampling a continuous-time signal $x_a(t)$ at uniform intervals of time



Discrete-Time Signals: Time-Domain Representation

- Here, n-th sample is given by

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT),\ n = \ldots, -2, -1, 0, 1, \ldots$$

- The spacing T between two consecutive samples is called the **sampling interval** or **sampling period**
- Reciprocal of sampling interval T, denoted as F_T, is called the **sampling frequency**:

$$F_T = \frac{1}{T}$$

Discrete-Time Signals: Time-Domain Representation

- Unit of sampling frequency is cycles per second, or **hertz** (Hz), if T is in seconds
- Whether or not the sequence $\{x[n]\}$ has been obtained by sampling, the quantity $x[n]$ is called the n**-th sample** of the sequence
- $\{x[n]\}$ is a **real sequence**, if the n-th sample $x[n]$ is real for all values of n
- Otherwise, $\{x[n]\}$ is a **complex sequence**

Discrete-Time Signals: Time-Domain Representation

- A complex sequence $\{x[n]\}$ can be written as $\{x[n]\} = \{x_{re}[n]\} + j\{x_{im}[n]\}$ where $x_{re}[n]$ and $x_{im}[n]$ are the real and imaginary parts of $x[n]$
- The complex conjugate sequence of $\{x[n]\}$ is given by $\{x^*[n]\} = \{x_{re}[n]\} - j\{x_{im}[n]\}$
- Often the braces are ignored to denote a sequence if there is no ambiguity

Discrete-Time Signals: Time-Domain Representation

- Example - $\{x[n]\} = \{\cos 0.25n\}$ is a real sequence
- $\{y[n]\} = \{e^{j0.3n}\}$ is a complex sequence
- We can write

$$\{y[n]\} = \{\cos 0.3n + j\sin 0.3n\}$$

$$= \{\cos 0.3n\} + j\{\sin 0.3n\}$$

where $\{y_{re}[n]\} = \{\cos 0.3n\}$

$$\{y_{im}[n]\} = \{\sin 0.3n\}$$

Discrete-Time Signals: Time-Domain Representation

- Example -

  $$\{w[n]\} = \{\cos 0.3n\} - j\{\sin 0.3n\} = \{e^{-j0.3n}\}$$

  is the complex conjugate sequence of $\{y[n]\}$

- That is,

  $$\{w[n]\} = \{y^*[n]\}$$

Discrete-Time Signals: Time-Domain Representation

- Two types of discrete-time signals:

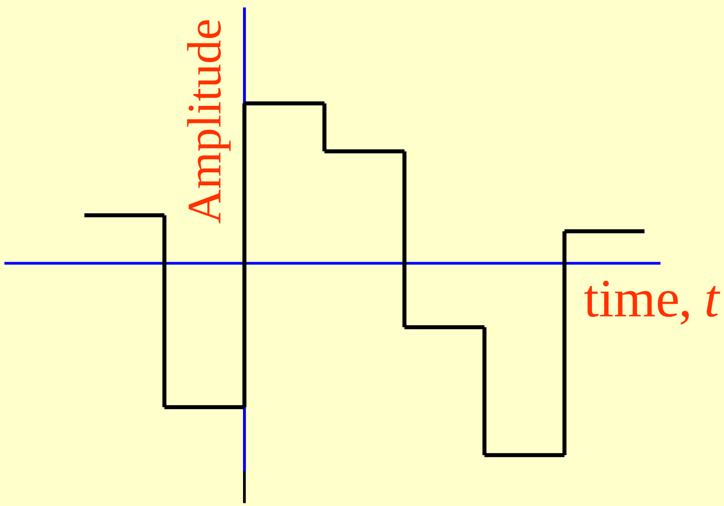
  - **Sampled-data signals** in which samples are continuous-valued

  - **Digital signals** in which samples are discrete-valued

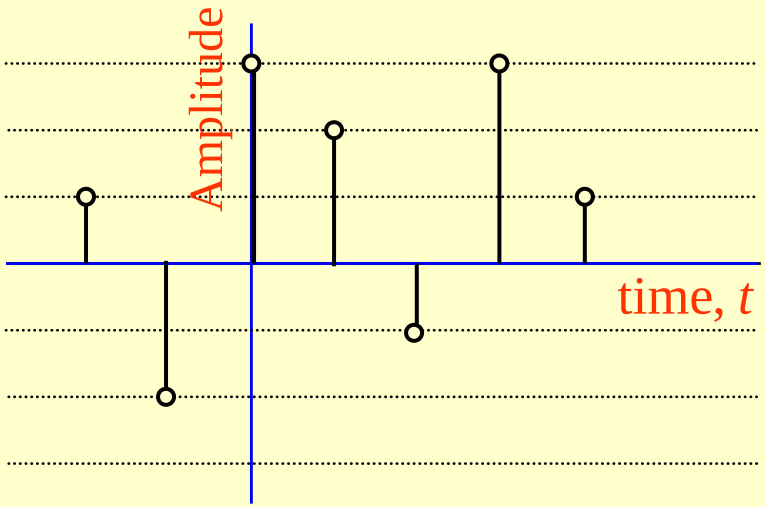
- Signals in a practical digital signal processing system are digital signals obtained by quantizing the sample values either by **rounding** or **truncation**

Discrete-Time Signals: Time-Domain Representation

- Example -



Boxedcar signal



Digital signal

Discrete-Time Signals: Time-Domain Representation

- A discrete-time signal may be a **finite-length** or an **infinite-length sequence**
- Finite-length (also called **finite-duration** or **finite-extent**) sequence is defined only for a finite time interval: $N_1 \leq n \leq N_2$

  where $-\infty < N_1$ and $N_2 < \infty$ with $N_1 \leq N_2$
- **Length** or **duration** of the above finite-length sequence is $N = N_2 - N_1 + 1$

Discrete-Time Signals: Time-Domain Representation

- Example - $x[n] = n^2, -3 \le n \le 4$ is a finite-length sequence of length $4-(-3)+1=8$

$y[n] = \cos 0.4n$ is an infinite-length sequence

Discrete-Time Signals: Time-Domain Representation

- A length-N sequence is often referred to as an N-**point sequence**
- The length of a finite-length sequence can be increased by **zero-padding**, i.e., by **appending** it **with zeros**

Discrete-Time Signals: Time-Domain Representation

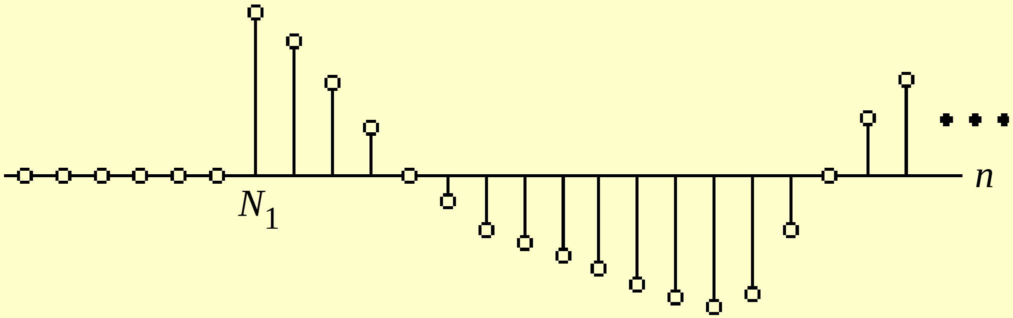
- Example -

$$x_e[n] = \begin{cases} n^2, & -3 \le n \le 4 \\ 0, & 5 \le n \le 8 \end{cases}$$

is a finite-length sequence of length 12 obtained by zero-padding $x[n] = n^2, -3 \le n \le 4$ with 4 zero-valued samples

Discrete-Time Signals: Time-Domain Representation

- A **right-sided sequence** $x[n]$ has zero-valued samples for $n < N_1$

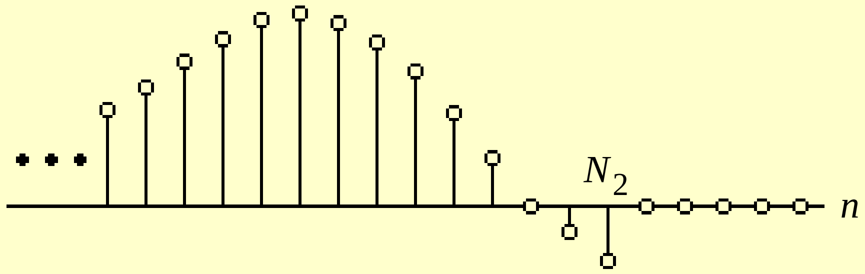


A right-sided sequence

- If $N_1 \geq 0,$ a right-sided sequence is called a **causal sequence**

Discrete-Time Signals: Time-Domain Representation

- A **left-sided sequence** $x[n]$ has zero-valued samples for $n > N_2$

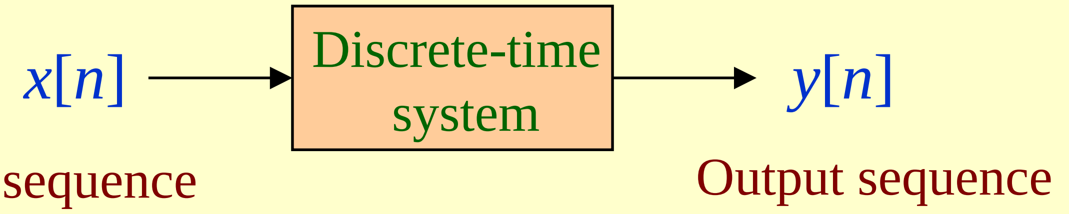


A left-sided sequence

- If $N_2 \leq 0$, a left-sided sequence is called a **anti-causal sequence**

Operations on Sequences

- A single-input, single-output discrete-time system operates on a sequence, called the **input sequence**, according some prescribed rules and develops another sequence, called the output sequence, with more desirable properties



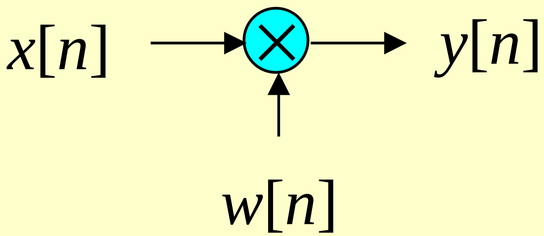
Operations on Sequences

- For example, the input may be a signal corrupted with additive noise
- Discrete-time system is designed to generate an output by removing the noise component from the input
- In most cases, the operation defining a particular discrete-time system is composed of some **basic operations**

Basic Operations

- **Product (modulation)** operation:

  – Modulator

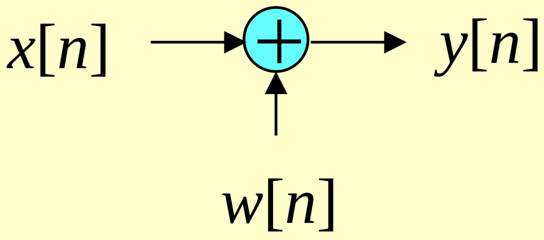


$$y[n] = x[n] \cdot w[n]$$

- An application is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called an **window sequence**
- Process called **windowing**

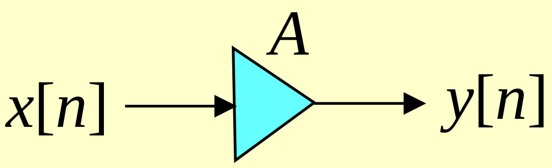
Basic Operations

- **Addition** operation:
  - Adder



$$y[n] = x[n] + w[n]$$

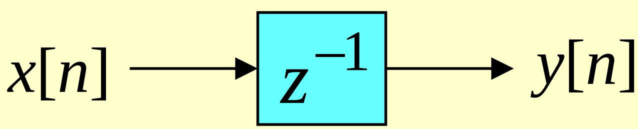
- **Multiplication** operation
  - Multiplier



$$y[n] = A \cdot x[n]$$

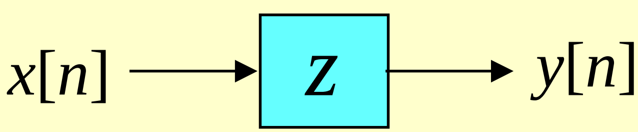
Basic Operations

- **Time-shifting** operation: $y[n] = x[n-N]$
  where N is an integer
- If $N > 0$, it is **delaying** operation
  - Unit delay



$$y[n] = x[n-1]$$

- If $N < 0$, it is an **advance** operation



$$y[n] = x[n+1]$$

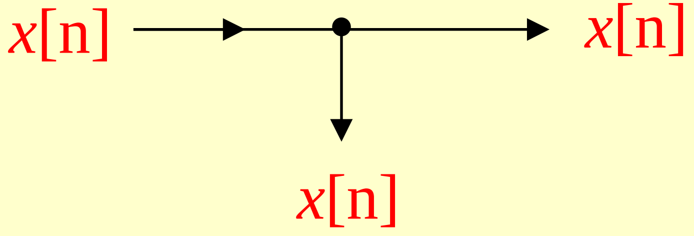
  - Unit advance

Basic Operations

- **Time-reversal (folding)** operation:

$$y[n] = x[-n]$$

- **Branching** operation: Used to provide multiple copies of a sequence



Basic Operations

- Example - Consider the two following sequences of length 5 defined for $0 \le n \le 4$:

$$\{a[n]\} = \{3 \quad 4 \quad 6 \quad -9 \quad 0\}$$

$$\{b[n]\} = \{2 \quad -1 \quad 4 \quad 5 \quad -3\}$$

- New sequences generated from the above two sequences by applying the basic operations are as follows:

Basic Operations

$$\{c[n]\} = \{a[n] \cdot b[n]\} = \{6 \quad -4 \quad 24 \quad -45 \quad 0\}$$

$$\{d[n]\} = \{a[n] + b[n]\} = \{5 \quad 3 \quad 10 \quad -4 \quad -3\}$$

$$\{e[n]\} = \frac{3}{2}\{a[n]\} = \{4.5 \quad 6 \quad 9 \quad -13.5 \quad 0\}$$

- As pointed out by the above example, operations on two or more sequences can be carried out if all sequences involved are of same length and defined for the same range of the time index n

Basic Operations

- However if the sequences are not of same length, in some situations, this problem can be circumvented by appending zero-valued samples to the sequence(s) of smaller lengths to make all sequences have the same range of the time index
- Example - Consider the sequence of length 3 defined for $0 \le n \le 2$: $\{f[n]\} = \{-2 \quad 1 \quad -3\}$

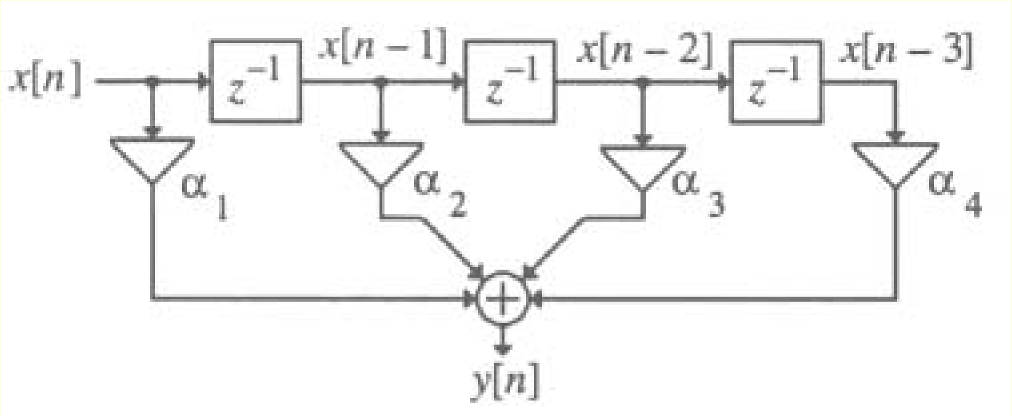
Basic Operations

- We cannot add the length-3 sequence $\{f[n]\}$ to the length-5 sequence $\{a[n]\}$ defined earlier
- We therefore first append $\{f[n]\}$ with 2 zero-valued samples resulting in a length-5 sequence $\{f_e[n]\} = \{-2 \quad 1 \quad -3 \quad 0 \quad 0\}$
- Then

$$\{g[n]\} = \{a[n]\} + \{f_e[n]\} = \{1 \quad 5 \quad 3 \quad -9 \quad 0\}$$

Combinations of Basic Operations

- Example -



$$y[n]=\alpha_1 x[n]+\alpha_2 x[n-1]+\alpha_3 x[n-2]+\alpha_4 x[n-3]$$

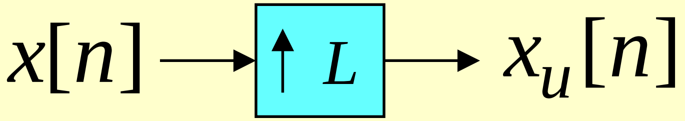
Sampling Rate Alteration

- Employed to generate a new sequence $y[n]$ with a sampling rate F_T' higher or lower than that of the sampling rate F_T of a given sequence $x[n]$
- **Sampling rate alteration ratio** is $R = \dfrac{F_T'}{F_T}$
- If $R > 1$, the process called **interpolation**
- If $R < 1$, the process called **decimation**

Sampling Rate Alteration

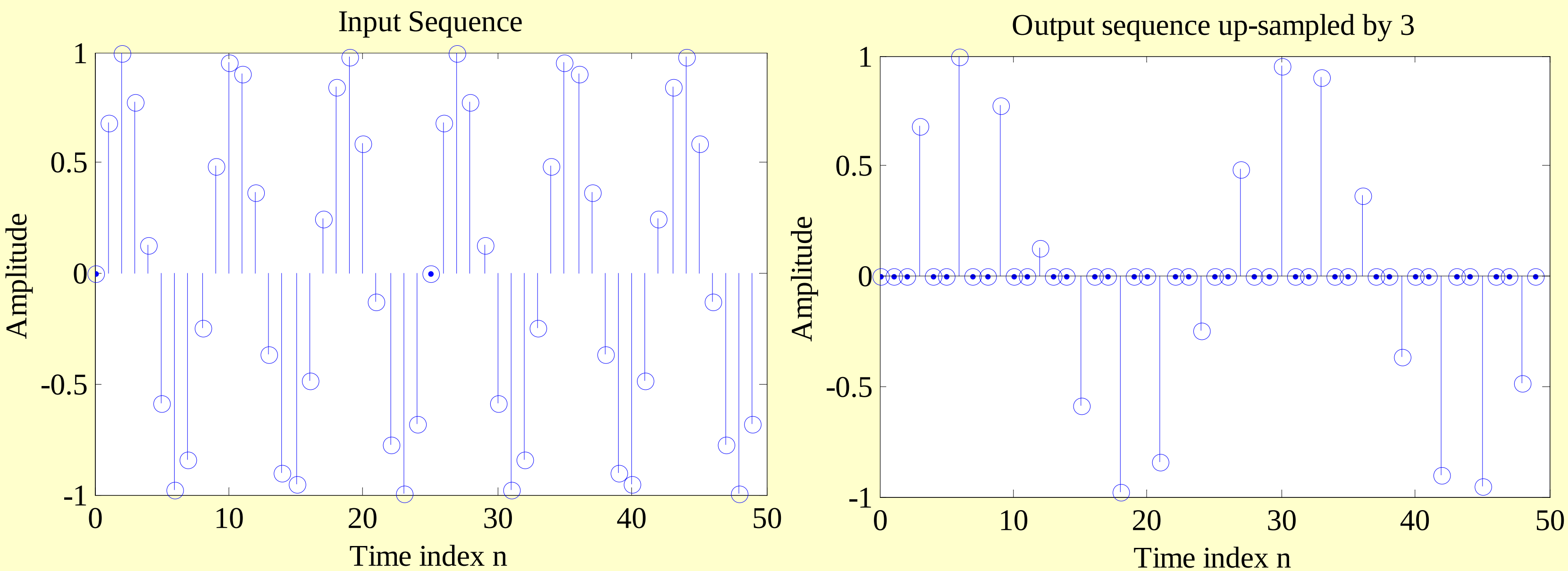
- In **up-sampling** by an integer factor $L > 1$, $L-1$ equidistant zero-valued samples are inserted by the **up-sampler** between each two consecutive samples of the input sequence $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$



Sampling Rate Alteration

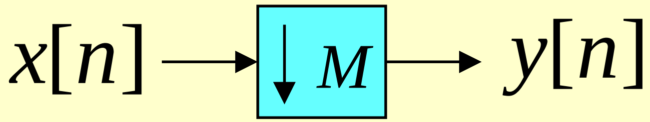
- An example of the up-sampling operation



Sampling Rate Alteration

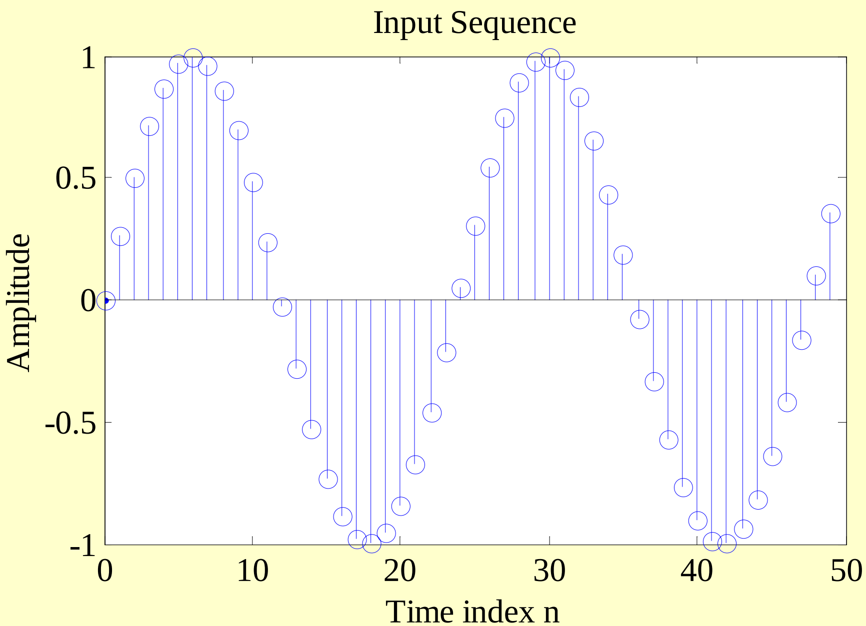
- In **down-sampling** by an integer factor $M > 1$, every M-th samples of the input sequence are kept and $M-1$ in-between samples are removed:

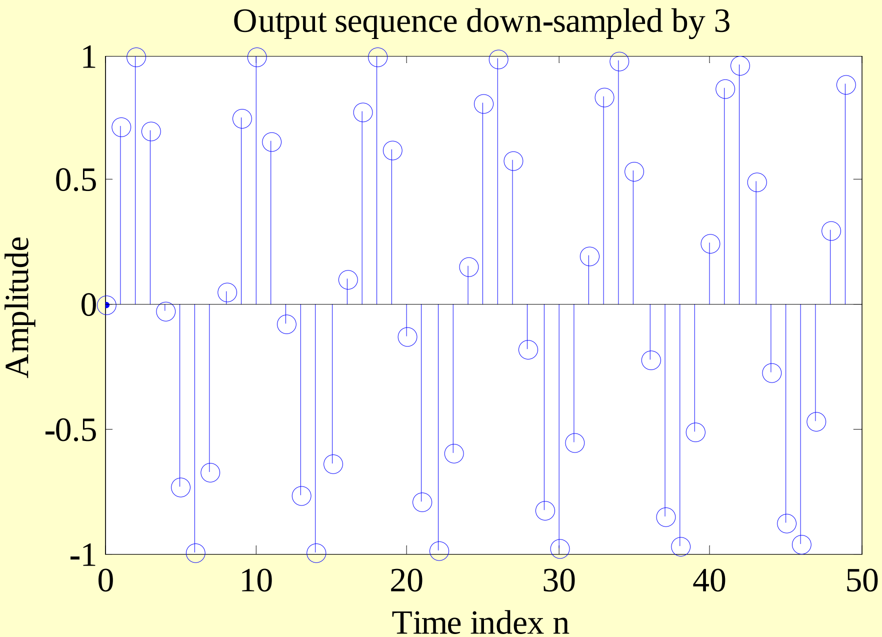
$$y[n] = x[nM]$$



Sampling Rate Alteration

- An example of the down-sampling operation



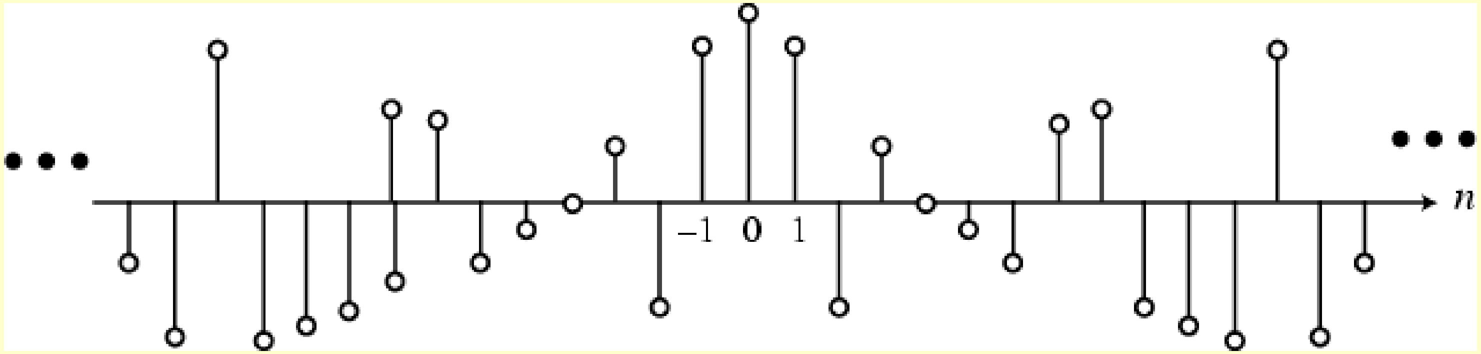


Classification of Sequences Based on Symmetry

- **Conjugate-symmetric sequence**:

$$x[n] = x^*[-n]$$

If $x[n]$ is real, then it is an **even sequence**



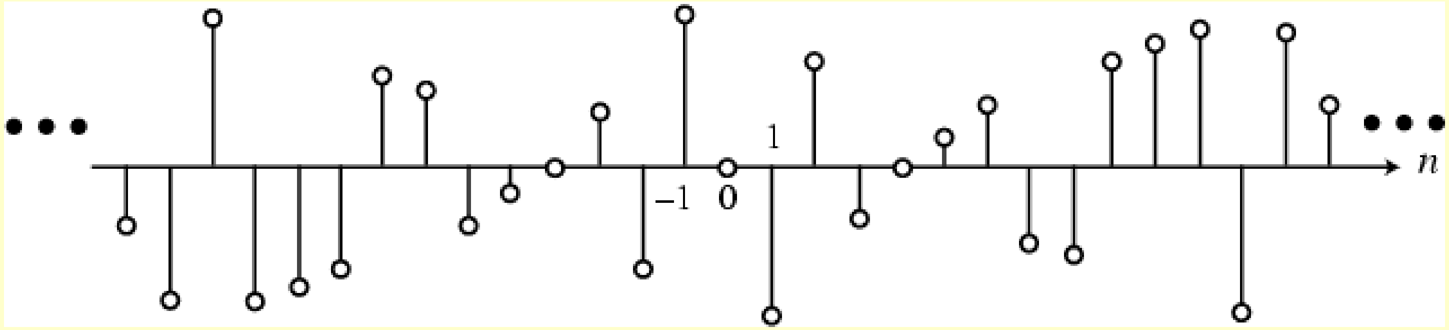
An even sequence

Classification of Sequences Based on Symmetry

- **Conjugate-antisymmetric sequence**:

$$x[n] = -x^*[-n]$$

If $x[n]$ is real, then it is an **odd sequence**



An odd sequence

Classification of Sequences Based on Symmetry

- It follows from the definition that for a conjugate-symmetric sequence $\{x[n]\}$, $x[0]$ must be a real number
- Likewise, it follows from the definition that for a conjugate anti-symmetric sequence $\{y[n]\}$, $y[0]$ must be an imaginary number
- From the above, it also follows that for an odd sequence $\{w[n]\}$, $w[0] = 0$

Classification of Sequences Based on Symmetry

- Any complex sequence can be expressed as a sum of its conjugate-symmetric part and its conjugate-antisymmetric part:

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$$

$$x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$$

Classification of Sequences Based on Symmetry

- <u>Example</u> - Consider the length-7 sequence defined for $-3 \leq n \leq 3$:

$$\{g[n]\} = \{0,\ 1+j4,\ -2+j3,\ \underset{\uparrow}{4-j2},\ -5-j6,\ -j2,\ 3\}$$

- Its conjugate sequence is then given by

$$\{g^*[n]\} = \{0,\ 1-j4,\ -2-j3,\ \underset{\uparrow}{4+j2},\ -5+j6,\ j2,\ 3\}$$

- The time-reversed version of the above is

$$\{g^*[-n]\} = \{3,\ j2,\ -5+j6,\ \underset{\uparrow}{4+j2},\ -2-j3,\ 1-j4,\ 0\}$$

Classification of Sequences Based on Symmetry

- Therefore $\{g_{cs}[n]\} = \frac{1}{2}\{g[n] + g^*[-n]\}$

$$= \{1.5,\ \ 0.5 + j3,\ \ -3.5 + j4.5,\ \ \underset{\uparrow}{4},\ \ -3.5 - j4.5,\ \ 0.5 - j3,\ \ 1.5\}$$

- Likewise $\{g_{ca}[n]\} = \frac{1}{2}\{g[n] - g^*[-n]\}$

$$= \{-1.5,\ \ 0.5 + j,\ \ 1.5 - j1.5,\ \ \underset{\uparrow}{-j2},\ \ -1.5 - j1.5,\ \ -0.5 - j,\ \ 1.5\}$$

- It can be easily verified that $g_{cs}[n] = g_{cs}^*[-n]$ and $g_{ca}[n] = -g_{ca}^*[-n]$

Classification of Sequences Based on Symmetry

- Any real sequence can be expressed as a sum of its even part and its odd part:

$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$x_{ev}[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_{od}[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$

Classification of Sequences Based on Symmetry

- A length-N sequence $x[n]$, $0 \leq n \leq N-1$, can be expressed as $x[n] = x_{pcs}[n] + x_{pca}[n]$ where

$$x_{pcs}[n] = \tfrac{1}{2}\left(x[n] + x^*[\langle -n \rangle_N]\right), \quad 0 \leq n \leq N-1,$$

is the **periodic conjugate-symmetric part** and

$$x_{pca}[n] = \tfrac{1}{2}\left(x[n] - x^*[\langle -n \rangle_N]\right), \quad 0 \leq n \leq N-1,$$

is the **periodic conjugate-antisymmetric part**

Classification of Sequences Based on Symmetry

- For a real sequence, the periodic conjugate-symmetric part, is a real sequence and is called the **periodic even part** $x_{pe}[n]$
- For a real sequence, the periodic conjugate-antisymmetric part, is a real sequence and is called the **periodic odd part** $x_{po}[n]$

Classification of Sequences Based on Symmetry

- A length-N sequence $x[n]$ is called a **periodic conjugate-symmetric sequence** if

$$x[n] = x^*[\langle -n \rangle_N] = x^*[N-n]$$

and is called a **periodic conjugate-antisymmetric sequence** if

$$x[n] = -x^*[\langle -n \rangle_N] = -x^*[N-n]$$

Classification of Sequences Based on Symmetry

- A finite-length real periodic conjugate-symmetric sequence is called a **symmetric sequence**
- A finite-length real periodic conjugate-antisymmetric sequence is called a **antisymmetric sequence**

Classification of Sequences Based on Symmetry

- Example - Consider the length-4 sequence defined for $0 \le n \le 3$:

  $$\{u[n]\} = \{1 + j4,\ \ -2 + j3,\ \ 4 - j2,\ \ -5 - j6\}$$

- Its conjugate sequence is given by

  $$\{u^*[n]\} = \{1 - j4,\ \ -2 - j3,\ \ 4 + j2,\ \ -5 + j6\}$$

- To determine the modulo-4 time-reversed version $\{u^*[\langle -n \rangle_4]\}$ observe the following:

Classification of Sequences Based on Symmetry

$$u^*[\langle -0\rangle_4] = u^*[0] = 1 - j4$$

$$u^*[\langle -1\rangle_4] = u^*[3] = -5 + j6$$

$$u^*[\langle -2\rangle_4] = u^*[2] = 4 + j2$$

$$u^*[\langle -3\rangle_4] = u^*[1] = -2 - j3$$

- Hence

$$\{u^*[\langle -n\rangle_4]\} = \{1 - j4,\ \ -5 + j6,\ \ 4 + j2,\ \ -2 - j3\}$$

Classification of Sequences Based on Symmetry

- Therefore

$$\{u_{pcs}[n]\} = \frac{1}{2}\{u[n] + u^*[\langle -n \rangle_4]\}$$

$$= \{1,\ -3.5 + j4.5,\ 4,\ -3.5 - j4.5\}$$

- Likewise

$$\{u_{pca}[n]\} = \frac{1}{2}\{u[n] - u^*[\langle -n \rangle_4]\}$$

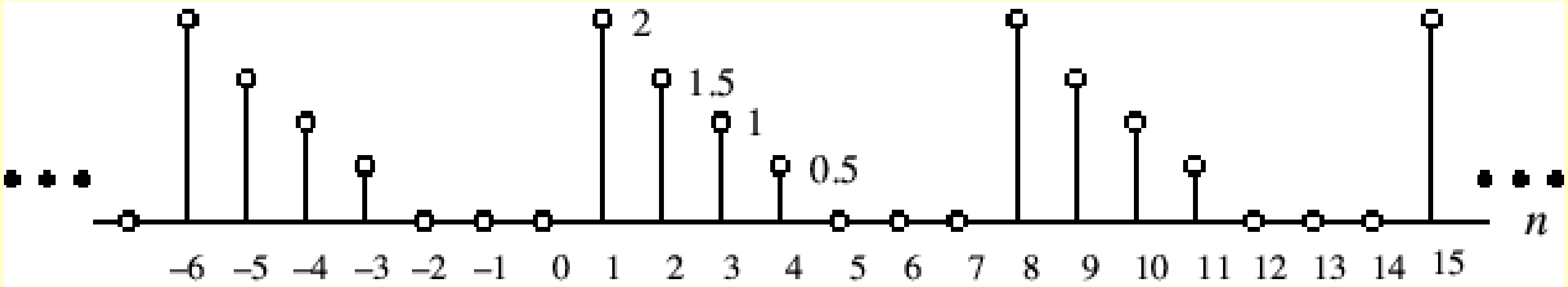
$$= \{j4,\ 1.5 - j1.5,\ -2,\ -1.5 - j1.5\}$$

Classification of Sequences Based on Periodicity

- A sequence $\tilde{x}[n]$ satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called a **periodic sequence** with a **period** N where N is a positive integer and k is any integer
- Smallest value of N satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called the **fundamental period**

Classification of Sequences Based on Periodicity

- Example -



- A sequence not satisfying the periodicity condition is called an **aperiodic sequence**

Classification of Sequences: Energy and Power Signals

- Total **energy** of a sequence $x[n]$ is defined by

$$\mathcal{E}_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

- An infinite length sequence with finite sample values may or may not have finite energy
- A finite length sequence with finite sample values has finite energy

Classification of Sequences: Energy and Power Signals

- The **average power** of an aperiodic sequence is defined by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} \left| x[n] \right|^2$$

- Define the **energy** of a sequence $x[n]$ over a finite interval $-K \le n \le K$ as

$$\mathcal{E}_{x,K} = \sum_{n=-K}^{K} \left| x[n] \right|^2$$

Classification of Sequences: Energy and Power Signals

- Then

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \mathcal{E}_{x.K}$$

- The **average power** of a periodic sequence $\tilde{x}[n]$ with a period N is given by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left|\tilde{x}[n]\right|^2$$

- The average power of an infinite-length sequence may be finite or infinite

Classification of Sequences: Energy and Power Signals

- Example - Consider the causal sequence defined by

$$x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Note: $x[n]$ has infinite energy
- Its average power is given by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1}\left(9\sum_{n=0}^{K} 1\right) = \lim_{K\to\infty} \frac{9(K+1)}{2K+1} = 4.5$$

Classification of Sequences: Energy and Power Signals

- An infinite energy signal with finite average power is called a **power signal**

  Example - A periodic sequence which has a finite average power but infinite energy

- A finite energy signal with zero average power is called an **energy signal**

  Example - A finite-length sequence which has finite energy but zero average power

Other Types of Classifications

- A sequence $x[n]$ is said to be **bounded** if

$$|x[n]| \le B_x < \infty$$

- Example - The sequence $x[n] = \cos 0.3\pi n$ is a bounded sequence as

$$|x[n]| = |\cos 0.3\pi n| \le 1$$

Other Types of Classifications

- A sequence $x[n]$ is said to be **absolutely summable** if
$$\sum_{n=-\infty}^{\infty} \left| x[n] \right| < \infty$$
- Example - The sequence
$$y[n] = \begin{cases} 0.3^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

is an absolutely summable sequence as
$$\sum_{n=0}^{\infty} \left| 0.3^n \right| = \frac{1}{1-0.3} = 1.42857 < \infty$$

Other Types of Classifications

- A sequence $x[n]$ is said to be **square-summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

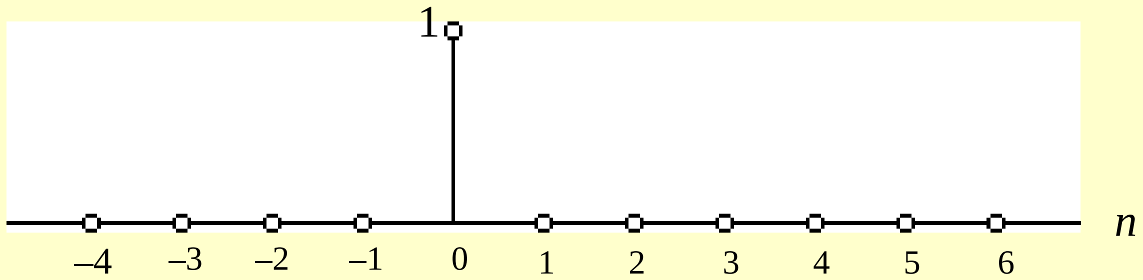
- Example - The sequence

$$h[n] = \frac{\sin 0.4n}{\pi n}$$

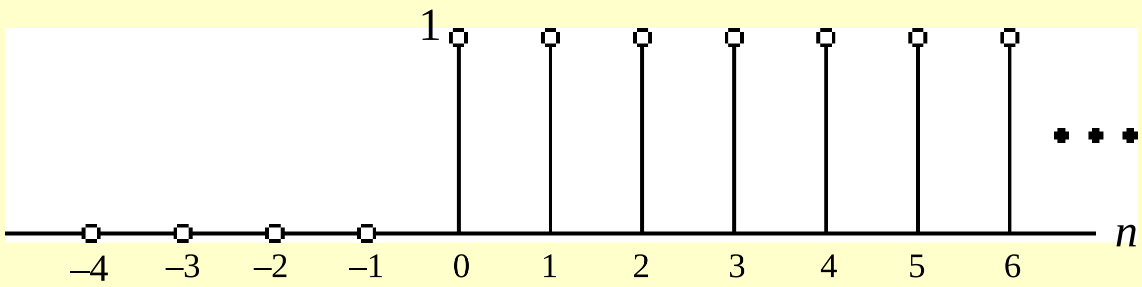
is square-summable but not absolutely summable

Basic Sequences

- **Unit sample sequence** - $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- **Unit step sequence** - $\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



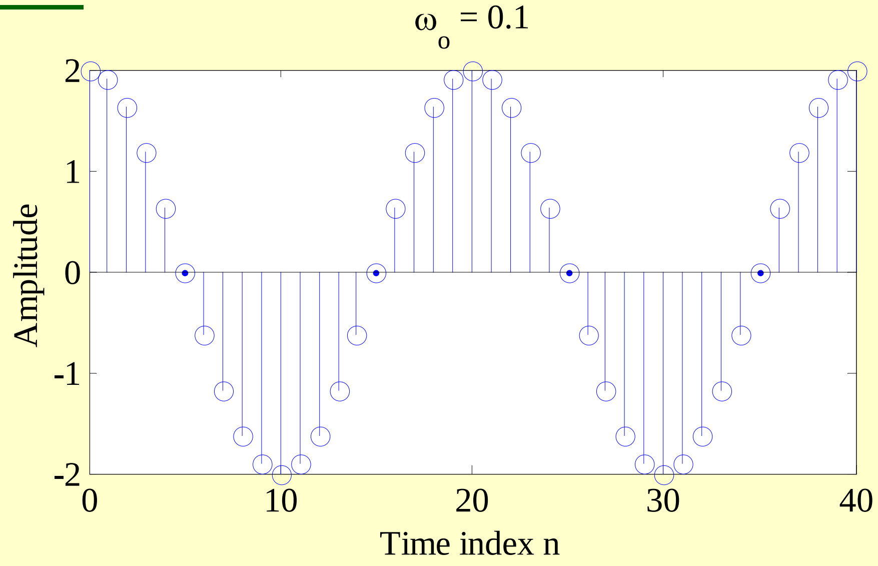
Basic Sequences

- **Real sinusoidal sequence** -

$$x[n] = A\cos(\omega_o n + \phi)$$

where A is the **amplitude**, ω_o is the **angular frequency**, and ϕ is the **phase** of $x[n]$

Example -



Basic Sequences

- **Exponential sequence -**

$$x[n] = A\alpha^n, \ -\infty < n < \infty$$

where A and α are real or complex numbers

- If we write $\alpha = e^{(\sigma_o + j\omega_o)}, A = |A| e^{j\phi}$,

then we can express

$$x[n] = |A| e^{j\phi} e^{(\sigma_o + j\omega_o)n} = x_{re}[n] + j\, x_{im}[n],$$

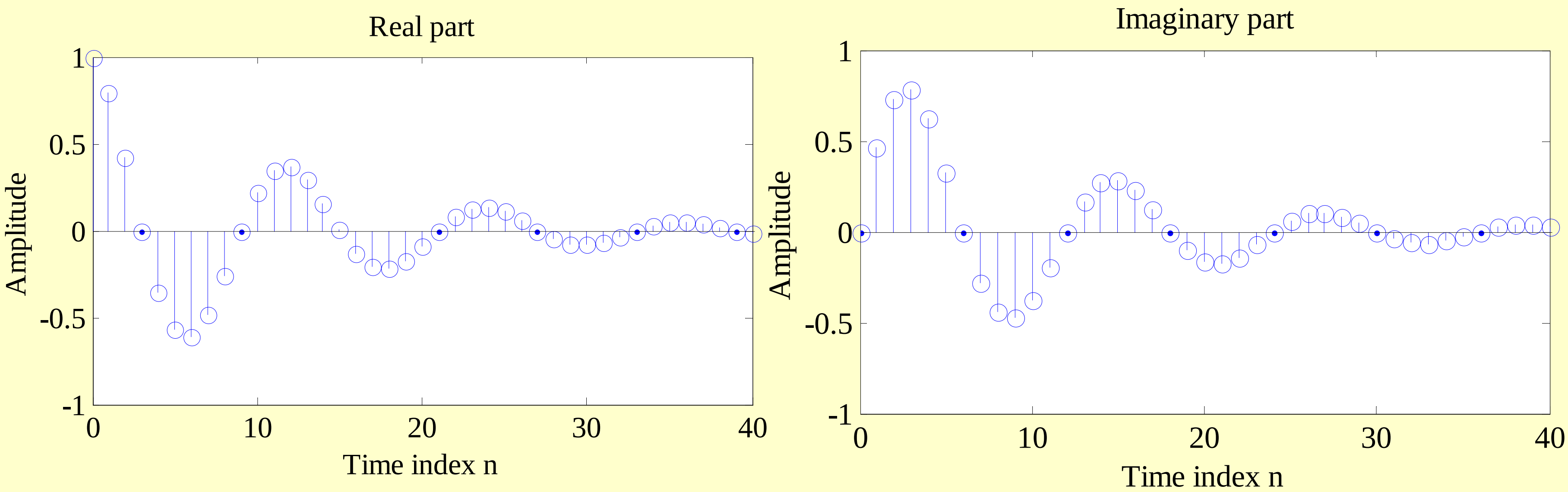
where

$$x_{re}[n] = |A| e^{\sigma_o n} \cos(\omega_o n + \phi),$$

$$x_{im}[n] = |A| e^{\sigma_o n} \sin(\omega_o n + \phi)$$

Basic Sequences

- $x_{re}[n]$ and $x_{im}[n]$ of a complex exponential sequence are real sinusoidal sequences with constant $(\sigma_o = 0)$, growing $(\sigma_o > 0)$, and decaying $(\sigma_o < 0)$ amplitudes for $n > 0$



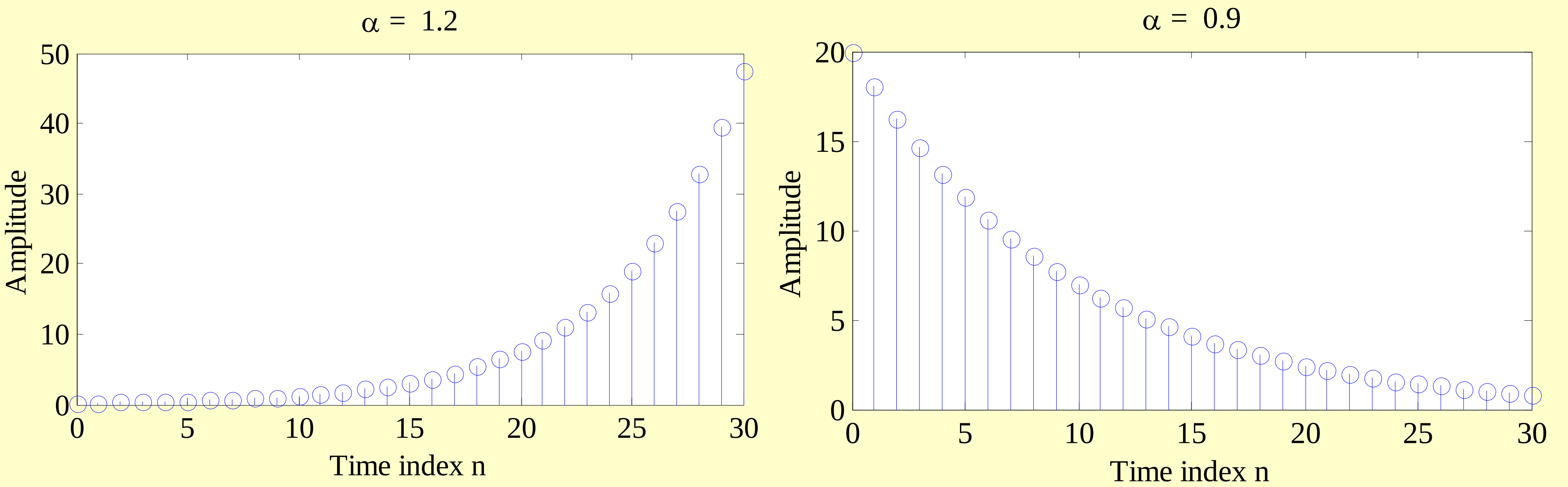
$$x[n] = \exp(-\frac{1}{12} + j\frac{\pi}{6})n$$

Basic Sequences

- **Real exponential sequence -**

$$x[n] = A\alpha^n, \; -\infty < n < \infty$$

where A and α are real numbers



Basic Sequences

- Sinusoidal sequence $A\cos(\omega_o n + \phi)$ and complex exponential sequence $B\exp(j\omega_o n)$ are periodic sequences of period N if $\omega_o N = 2\pi r$ where N and r are positive integers
- Smallest value of N satisfying $\omega_o N = 2\pi r$ is the **fundamental period** of the sequence
- To verify the above fact, consider

$$x_1[n] = \cos(\omega_o n + \phi)$$
$$x_2[n] = \cos(\omega_o (n + N) + \phi)$$

Basic Sequences

- Now $x_2[n] = \cos(\omega_o(n+N)+\phi)$

  $= \cos(\omega_o n+\phi)\cos\omega_o N - \sin(\omega_o n+\phi)\sin\omega_o N$

  which will be equal to $\cos(\omega_o n+\phi) = x_1[n]$
  only if

  $\sin\omega_o N = 0$ and $\cos\omega_o N = 1$

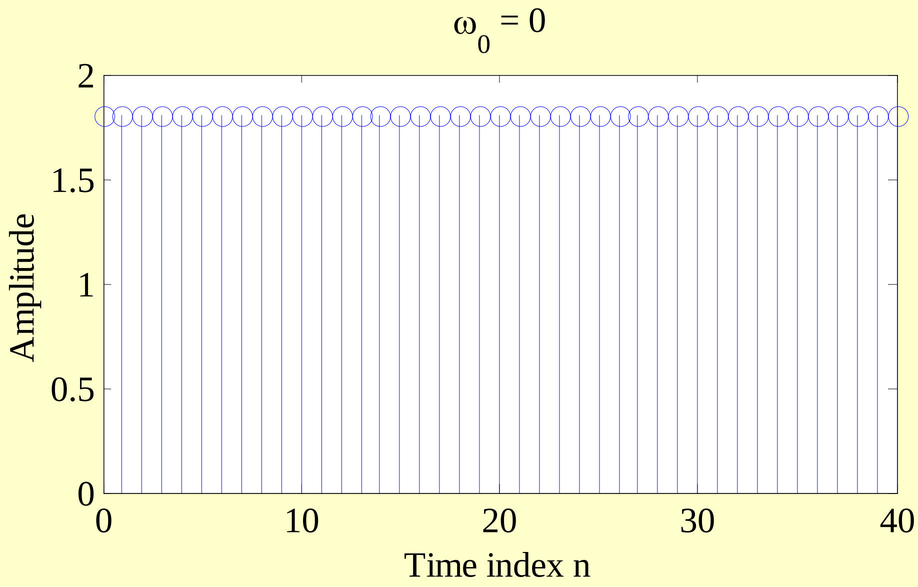
- These two conditions are met if and only if

  $\omega_o N = 2\pi r$ or $\dfrac{2\pi}{\omega_o} = \dfrac{N}{r}$

Basic Sequences

- If $2\pi/\omega_o$ is a noninteger rational number, then the period will be a multiple of $2\pi/\omega_o$
- Otherwise, the sequence is **aperiodic**
- Example - $x[n] = \sin(\sqrt{3}n + \phi)$ is an aperiodic sequence

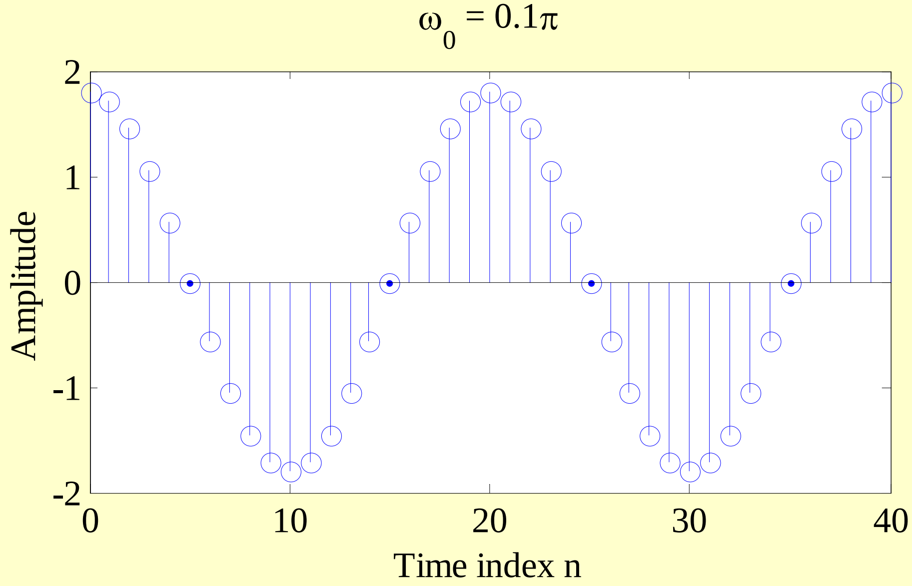
Basic Sequences



- Here $\omega_o = 0$

- Hence period $N = \dfrac{2\pi r}{0} = 1$ for $r = 0$

Basic Sequences



- Here $\omega_o = 0.1\pi$

- Hence $N = \dfrac{2\pi r}{0.1\pi} = 20$ for $r = 1$

Basic Sequences

- Property 1 - Consider $x[n] = \exp(j\omega_1 n)$ and $y[n] = \exp(j\omega_2 n)$ with $0 \leq \omega_1 < \pi$ and $2\pi k \leq \omega_2 < 2\pi(k+1)$ where k is any positive integer
- If $\omega_2 = \omega_1 + 2\pi k$, then $x[n] = y[n]$

- Thus, $x[n]$ and $y[n]$ are indistinguishable

Basic Sequences

- Property 2 - The frequency of oscillation of $A\cos(\omega_o n)$ increases as ω_o increases from 0 to π, and then decreases as ω_o increases from π to 2π
- Thus, frequencies in the neighborhood of $\omega = 0$ are called **low frequencies**, whereas, frequencies in the neighborhood of $\omega = \pi$ are called **high frequencies**

Basic Sequences

- Because of Property 1, a frequency ω_o in the neighborhood of $\omega = 2\pi$ k is indistinguishable from a frequency $\omega_o - 2\pi k$ in the neighborhood of $\omega = 0$

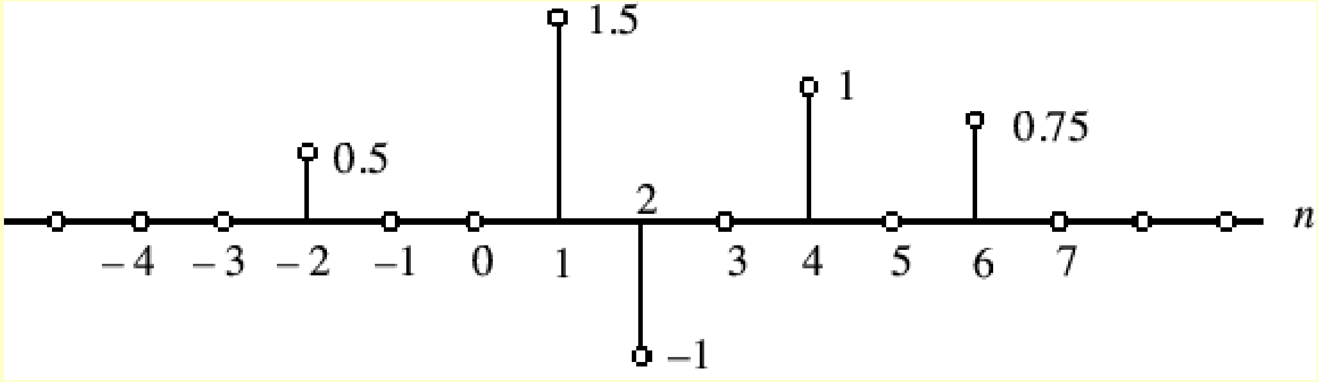
  and a frequency ω_o in the neighborhood of $\omega = \pi(2k+1)$ is indistinguishable from a frequency $\omega_o - \pi(2k+1)$ in the neighborhood of $\omega = \pi$

Basic Sequences

- Frequencies in the neighborhood of $\omega = 2\pi k$ are usually called **low frequencies**
- Frequencies in the neighborhood of $\omega = \pi(2k+1)$ are usually called **high frequencies**
- $v_1[n] = \cos(0.1\pi n) = \cos(1.9\pi n)$ is a low-frequency signal
- $v_2[n] = \cos(0.8\pi n) = \cos(1.2\pi n)$ is a high-frequency signal

Basic Sequences

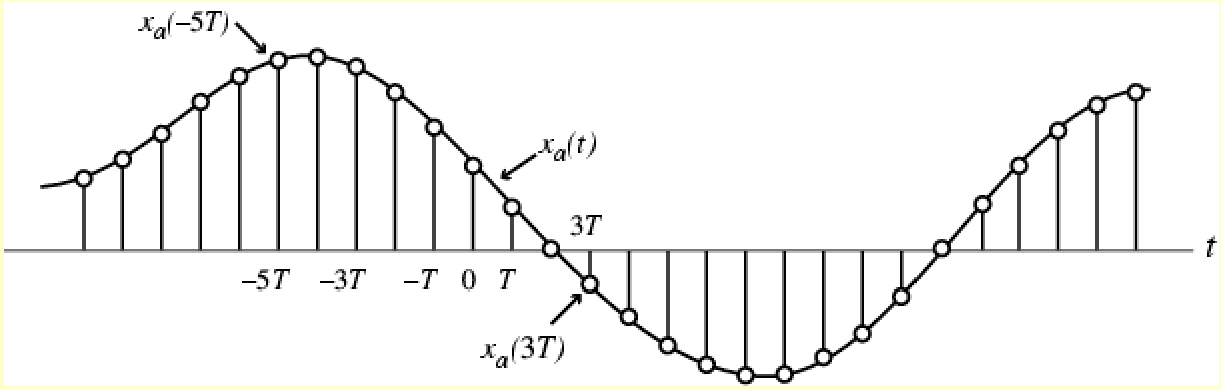
- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions



$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\delta[n-6]$$

The Sampling Process

- Often, a discrete-time sequence $x[n]$ is developed by uniformly sampling a continuous-time signal $x_a(t)$ as indicated below



- The relation between the two signals is

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT),\; n = \ldots, -2, -1, 0, 1, 2, \ldots$$

The Sampling Process

- Time variable t of $x_a(t)$ is related to the time variable n of $x[n]$ only at discrete-time instants t_n given by

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

with $F_T = 1/T$ denoting the sampling frequency and

$\Omega_T = 2\pi F_T$ denoting the sampling angular frequency

The Sampling Process

- Consider the continuous-time signal

$$x(t) = A\cos(2\pi f_o t + \phi) = A\cos(\Omega_o t + \phi)$$

- The corresponding discrete-time signal is

$$x[n] = A\cos(\Omega_o nT + \phi) = A\cos(\frac{2\pi\Omega_o}{\Omega_T} n + \phi)$$

$$= A\cos(\omega_o n + \phi)$$

where $\omega_o = 2\pi\Omega_o / \Omega_T = \Omega_o T$

is the normalized digital angular frequency of $x[n]$

The Sampling Process

- If the unit of sampling period T is in seconds
- The unit of normalized digital angular frequency ω_o is radians/sample
- The unit of normalized analog angular frequency Ω_o is radians/second
- The unit of analog frequency f_o is hertz (Hz)

The Sampling Process

- The three continuous-time signals

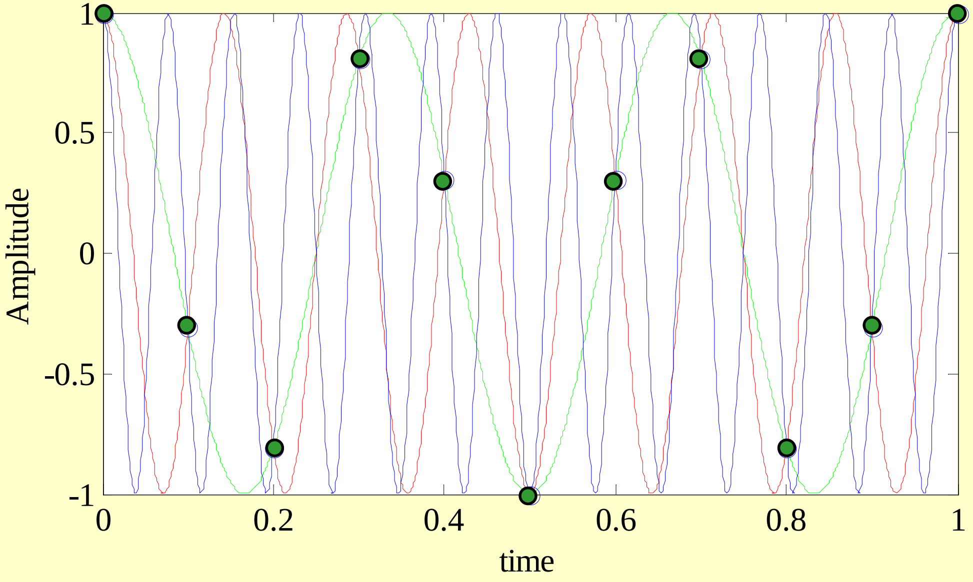
$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

of frequencies 3 Hz, 7 Hz, and 13 Hz, are sampled at a sampling rate of 10 Hz, i.e. with $T = 0.1$ sec. generating the three sequences

$$g_1[n] = \cos(0.6\pi n) \qquad g_2[n] = \cos(1.4\pi n)$$
$$g_3[n] = \cos(2.6\pi n)$$

The Sampling Process

- Plots of these sequences (shown with circles) and their parent time functions are shown below:



- Note that each sequence has exactly the same sample value for any given n

The Sampling Process

- This fact can also be verified by observing that

$$g_2[n] = \cos(1.4\pi n) = \cos\big((2\pi - 0.6\pi)n\big) = \cos(0.6\pi n)$$

$$g_3[n] = \cos(2.6\pi n) = \cos\big((2\pi + 0.6\pi)n\big) = \cos(0.6\pi n)$$

- As a result, all three sequences are identical and it is difficult to associate a unique continuous-time function with each of these sequences

The Sampling Process

- The above phenomenon of a continuous-time signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called **aliasing**

The Sampling Process

- Since there are an infinite number of continuous-time signals that can lead to the same sequence when sampled periodically, additional conditions need to imposed so that the sequence $\{x[n]\} = \{x_a(nT)\}$ can uniquely represent the parent continuous-time signal $x_a(t)$
- In this case, $x_a(t)$ can be fully recovered from $\{x[n]\}$

The Sampling Process

- Example - Determine the discrete-time signal $v[n]$ obtained by uniformly sampling at a sampling rate of 200 Hz the continuous-time signal

$$v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t)$$

- Note: $v_a(t)$ is composed of 5 sinusoidal signals of frequencies 30 Hz, 150 Hz, 170 Hz, 250 Hz and 330 Hz

The Sampling Process

- The sampling period is $T = \frac{1}{200} = 0.005 \sec$
- The generated discrete-time signal $v[n]$ is thus given by

$$v[n] = 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n)$$
$$+ 4\cos(2.5\pi n) + 10\sin(3.3\pi n)$$
$$= 6\cos(0.3\pi n) + 3\sin\big((2\pi - 0.5\pi)n\big) + 2\cos\big((2\pi - 0.3\pi)n\big)$$
$$+ 4\cos\big((2\pi + 0.5\pi)n\big) + 10\sin\big((4\pi - 0.7\pi)n\big)$$

The Sampling Process

$$= 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) + 4\cos(0.5\pi n)$$
$$- 10\sin(0.7\pi n)$$

$$= 8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)$$

- Note: $v[n]$ is composed of 3 discrete-time sinusoidal signals of normalized angular frequencies: 0.3π, 0.5π, and 0.7π

The Sampling Process

- Note: An identical discrete-time signal is also generated by uniformly sampling at a 200-Hz sampling rate the following continuous-time signals:

$$w_a(t) = 8\cos(60\pi t) + 5\cos(100\pi t + 0.6435) - 10\sin(140\pi t)$$

$$g_a(t) = 2\cos(60\pi t) + 4\cos(100\pi t) + 10\sin(260\pi t) + 6\cos(460\pi t) + 3\sin(700\pi t)$$

The Sampling Process

- Recall $\omega_o = \dfrac{2\pi\Omega_o}{\Omega_T}$
- Thus if $\Omega_T > 2\Omega_o$, then the corresponding normalized digital angular frequency ω_o of the discrete-time signal obtained by sampling the parent continuous-time sinusoidal signal will be in the range $-\pi < \omega < \pi$
- No aliasing

The Sampling Process

- On the other hand, if $\Omega_T < 2\Omega_o$, the normalized digital angular frequency will foldover into a lower digital frequency $\omega_o = \langle 2\pi\Omega_o / \Omega_T \rangle_{2\pi}$ in the range $-\pi < \omega < \pi$ because of aliasing
- Hence, to prevent aliasing, the sampling frequency Ω_T should be greater than 2 times the frequency Ω_o of the sinusoidal signal being sampled

The Sampling Process

- Generalization: Consider an arbitrary continuous-time signal $x_a(t)$ composed of a weighted sum of a number of sinusoidal signals
- $x_a(t)$ can be represented uniquely by its sampled version $\{x[n]\}$ if the sampling frequency Ω_T is chosen to be greater than 2 times the highest frequency contained in $x_a(t)$

The Sampling Process

- The condition to be satisfied by the sampling frequency to prevent aliasing is called the **sampling theorem**
- A formal proof of this theorem will be presented later