Digital Filter Design

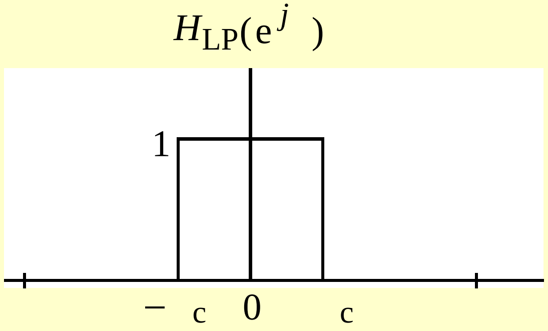
- Objective - Determination of a realizable transfer function $G(z)$ approximating a given frequency response specification is an important step in the development of a digital filter
- If an IIR filter is desired, $G(z)$ should be a stable real rational function
- Digital filter design is the process of deriving the transfer function $G(z)$

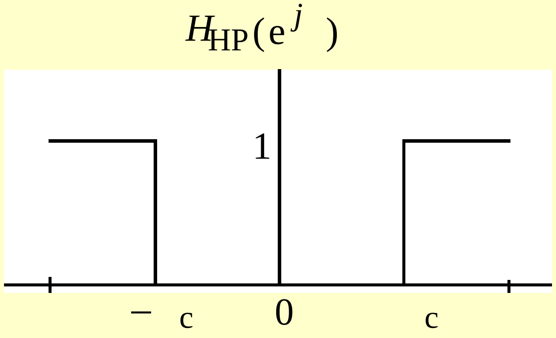
Digital Filter Specifications

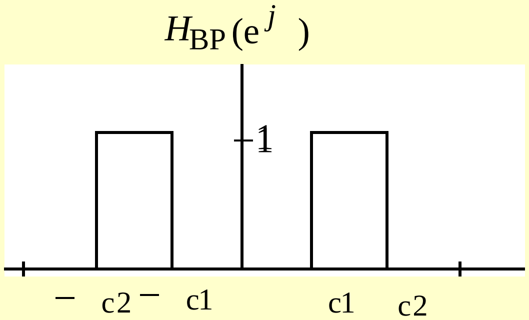
- Usually, either the magnitude and/or the phase (delay) response is specified for the design of digital filter for most applications
- In some situations, the unit sample response or the step response may be specified
- In most practical applications, the problem of interest is the development of a realizable approximation to a given magnitude response specification

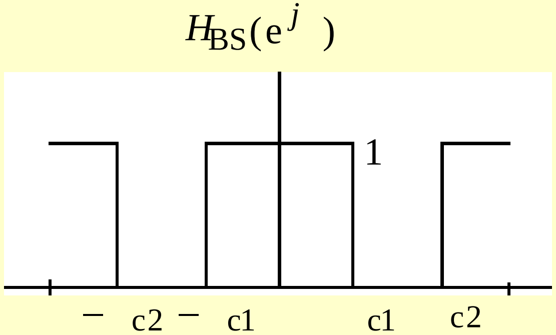
Digital Filter Specifications

- We discuss in this course only the magnitude approximation problem
- There are four basic types of ideal filters with magnitude responses as shown below







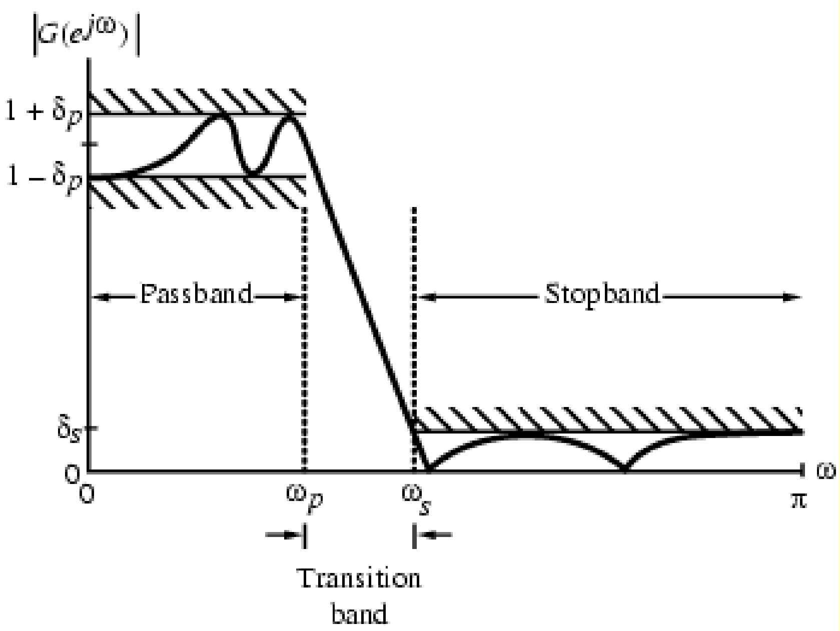


Digital Filter Specifications

- As the impulse response corresponding to each of these ideal filters is noncausal and of infinite length, these filters are not realizable
- In practice, the magnitude response specifications of a digital filter in the passband and in the stopband are given with some acceptable tolerances
- In addition, a transition band is specified between the passband and stopband

Digital Filter Specifications

- For example, the magnitude response $\left|G(e^{j\omega})\right|$ of a digital lowpass filter may be given as indicated below



Digital Filter Specifications

- As indicated in the figure, in the **passband**, defined by $0 \le \omega \le \omega_p$, we require that $\left|G(e^{j\omega})\right| \cong 1$ with an error $\pm\delta_p$, i.e.,

$$1-\delta_p \le \left|G(e^{j\omega})\right| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- In the **stopband**, defined by $\omega_s \le \omega \le \pi$, we require that $\left|G(e^{j\omega})\right| \cong 0$ with an error δ_s, i.e.,

$$\left|G(e^{j\omega})\right| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

Digital Filter Specifications

- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- Since $G(e^{j\omega})$ is a periodic function of ω, and $|G(e^{j\omega})|$ of a real-coefficient digital filter is an even function of ω
- As a result, filter specifications are given only for the frequency range $0 \leq |\omega| \leq \pi$

Digital Filter Specifications

- Specifications are often given in terms of **loss function** $\mathrm{G}(\omega) = -20\log_{10}\left|G(e^{j\omega})\right|$ in dB
- **Peak passband ripple**

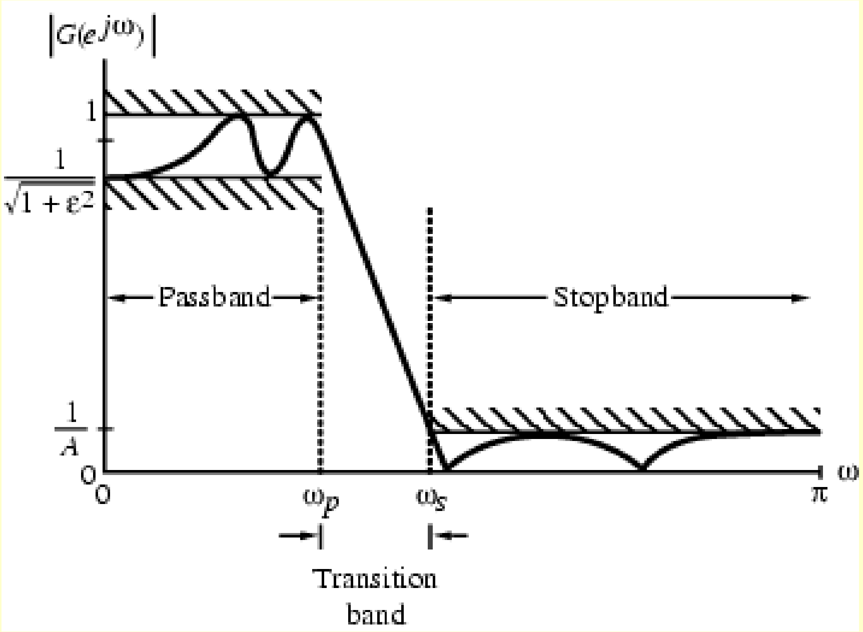
$$\alpha_p = -20\log_{10}(1-\delta_p)\ \mathrm{dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s)\ \mathrm{dB}$$

Digital Filter Specifications

- Magnitude specifications may alternately be given in a normalized form as indicated below



Digital Filter Specifications

- Here, the maximum value of the magnitude in the passband is assumed to be unity

- $1/\sqrt{1+\varepsilon^2}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband

- $\dfrac{1}{A}$ - Maximum stopband magnitude

Digital Filter Specifications

- For the normalized specification, maximum value of the gain function or the minimum value of the loss function is 0 dB
- **Maximum passband attenuation** -

$$\alpha_{\max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right) \text{ dB}$$

- For $\delta_p << 1$, it can be shown that

$$\alpha_{\max} \cong -20\log_{10}(1-2\delta_p) \text{ dB}$$

Digital Filter Specifications

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

Digital Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then

$$\omega_p = \frac{2\pi(7\times10^3)}{25\times10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times10^3)}{25\times10^3} = 0.24\pi$$

Selection of Filter Type

- The transfer function $H(z)$ meeting the frequency response specifications should be a causal transfer function
- For IIR digital filter design, the IIR transfer function is a real rational function of z^{-1}:

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}$$

- $H(z)$ must be a stable transfer function and must be of lowest order N for reduced computational complexity

Selection of Filter Type

- For FIR digital filter design, the FIR transfer function is a polynomial in z^{-1} with real coefficients:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree N of $H(z)$ must be as small as possible
- If a linear phase is desired, the filter coefficients must satisfy the constraint:

$$h[n] = \pm h[N-n]$$

Selection of Filter Type

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase,

  (2) Filter structure always stable with quantized coefficients

- Disadvantages in using an FIR filter - Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

Digital Filter Design: Basic Approaches

- Most common approach to IIR filter design - (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications
- (2) Determine the analog lowpass filter transfer function $H_a(s)$
- (3) Transform $H_a(s)$ into the desired digital transfer function $G(z)$

Digital Filter Design: Basic Approaches

- This approach has been widely used for the following reasons:

  (1) Analog approximation techniques are highly advanced

  (2) They usually yield closed-form solutions

  (3) Extensive tables are available for analog filter design

  (4) Many applications require digital simulation of analog systems

Digital Filter Design: Basic Approaches

- An analog transfer function to be denoted as

$$H_a(s) = \frac{P_a(s)}{D_a(s)}$$

where the subscript “a” specifically indicates the analog domain

- A digital transfer function derived from $H_a(s)$ shall be denoted as

$$G(z) = \frac{P(z)}{D(z)}$$

Digital Filter Design: Basic Approaches

- Basic idea behind the conversion of $H_a(s)$ into $G(z)$ is to apply a mapping from the *s*-domain to the *z*-domain so that essential properties of the analog frequency response are preserved
- Thus mapping function should be such that
  - Imaginary $(j\Omega)$ axis in the *s*-plane be mapped onto the unit circle of the *z*-plane
  - A stable analog transfer function be mapped into a stable digital transfer function

Digital Filter Design: Basic Approaches

- FIR filter design is based on a direct approximation of the specified magnitude response, with the often added requirement that the phase be linear
- The design of an FIR filter of order N may be accomplished by finding either the length-(N+1) impulse response samples $\{h[n]\}$ or the (N+1) samples of its frequency response $H(e^{j\omega})$

Digital Filter Design: Basic Approaches

- Three commonly used approaches to FIR filter design -

  (1) Windowed Fourier series approach

  (2) Frequency sampling approach

  (3) Computer-based optimization methods

IIR Digital Filter Design: Bilinear Transformation Method

- Bilinear transformation -

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

- Above transformation maps a single point in the *s*-plane to a unique point in the *z*-plane and vice-versa
- Relation between $G(z)$ and $H_a(s)$ is then given by

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Bilinear Transformation

- Digital filter design consists of 3 steps:

  (1) Develop the specifications of $H_a(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$

  (2) Design $H_a(s)$

  (3) Determine $G(z)$ by applying bilinear transformation to $H_a(s)$

- As a result, the parameter T has no effect on $G(z)$ and $T = 2$ is chosen for convenience

Bilinear Transformation

- Inverse bilinear transformation for $T = 2$ is

$$z = \frac{1+s}{1-s}$$

- For $s = \sigma_o + j\Omega_o$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \quad \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

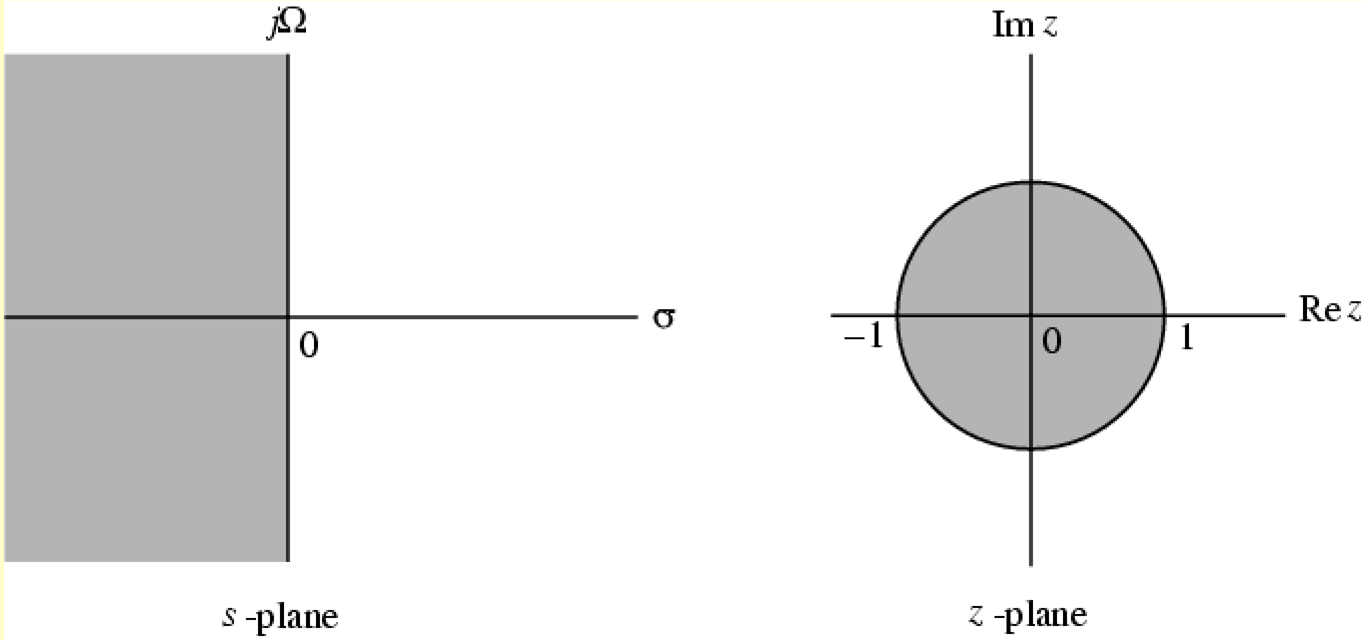
- Thus, $\sigma_o = 0 \rightarrow |z| = 1$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

Bilinear Transformation

- Mapping of *s*-plane into the *z*-plane



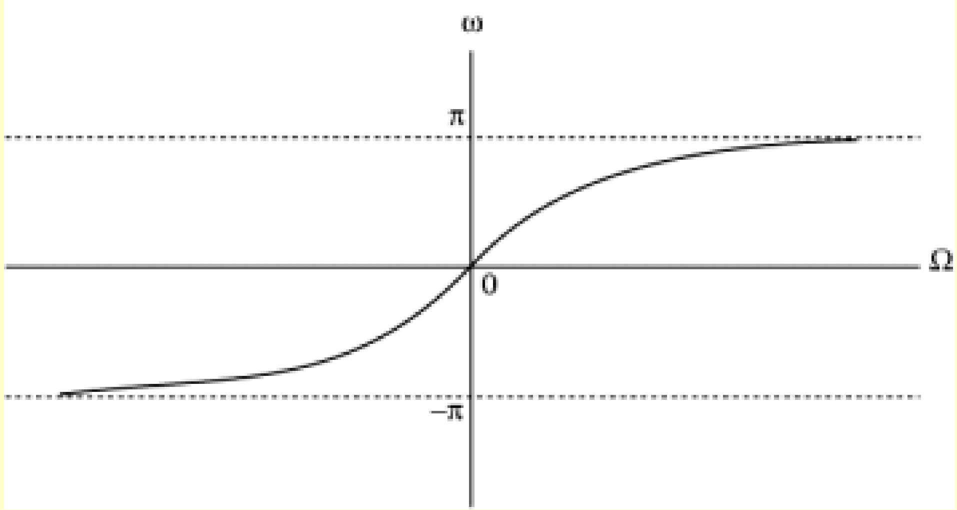
Bilinear Transformation

- For $z = e^{j\omega}$ with $T = 2$ we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})}$$

$$= \frac{j2\sin(\omega/2)}{2\cos(\omega/2)} = 2\tan(\omega/2)$$

or $\Omega = \tan(\omega/2)$

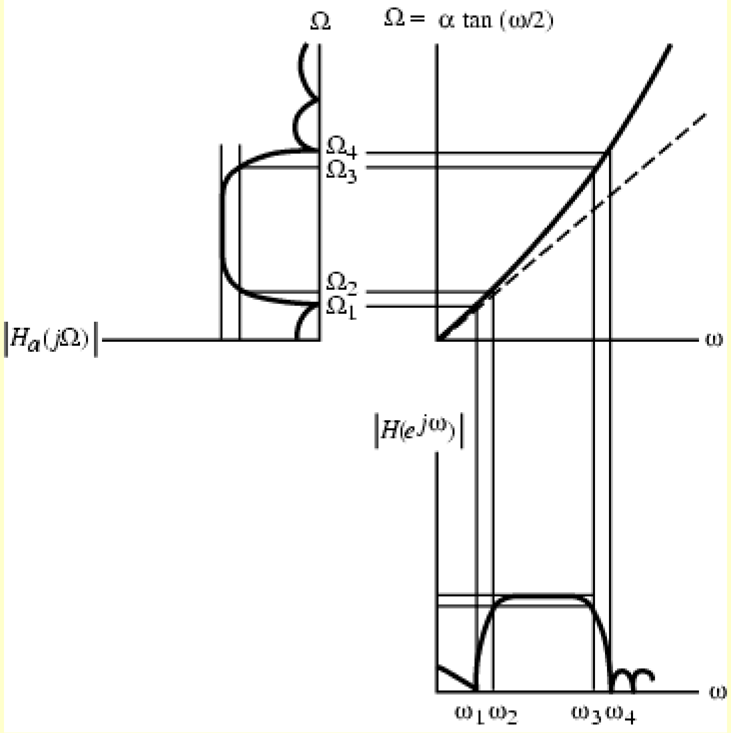


Bilinear Transformation

- Mapping is highly nonlinear
- Complete negative imaginary axis in the *s*-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle in the *z*-plane from $z = -1$ to $z = 1$
- Complete positive imaginary axis in the *s*-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the *z*-plane from $z = 1$ to $z = -1$

Bilinear Transformation

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- Effect of warping shown below



Bilinear Transformation

- Steps in the design of a digital filter -

  (1) Prewarp (ω_p, ω_s) to find their analog equivalents (Ω_p, Ω_s)

  (2) Design the analog filter $H_a(s)$

  (3) Design the digital filter $G(z)$ by applying bilinear transformation to $H_a(s)$

- Transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values
- Transformation does not preserve phase response of analog filter

IIR Digital Filter Design Using Bilinear Transformation

- Example - Consider

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

- Applying bilinear transformation to the above we get the transfer function of a first-order digital lowpass Butterworth filter

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1}) + \Omega_c(1+z^{-1})}$$

IIR Digital Filter Design Using Bilinear Transformation

- Rearranging terms we get

$$G(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}}$$

where

$$\alpha = \frac{1-\Omega_c}{1+\Omega_c} = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}$$

IIR Digital Filter Design Using Bilinear Transformation

- Example - Consider the second-order analog notch transfer function

$$H_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + B\,s + \Omega_o^2}$$

for which $|H_a(j\Omega_o)| = 0$

$$|H_a(j0)| = |H_a(j\infty)| = 1$$

- Ω_o is called the **notch frequency**
- If $|H_a(j\Omega_2)| = |H_a(j\Omega_1)| = 1/\sqrt{2}$ then $B = \Omega_2 - \Omega_1$ is the **3-dB notch bandwidth**

IIR Digital Filter Design Using Bilinear Transformation

- Then $$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{(1+\Omega_o^2) - 2(1-\Omega_o^2)\,z^{-1} + (1+\Omega_o^2)\,z^{-2}}{(1+\Omega_o^2+B) - 2(1-\Omega_o^2)\,z^{-1} + (1+\Omega_o^2-B)\,z^{-2}}$$

$$= \frac{1+\alpha}{2}\cdot\frac{1-2\beta\,z^{-1}+z^{-2}}{1-2\beta(1+\alpha)z^{-1}+\alpha\,z^{-2}}$$

where $$\alpha = \frac{1+\Omega_o^2-B}{1+\Omega_o^2+B} = \frac{1-\tan(B_w/2)}{1+\tan(B_w/2)}$$

$$\beta = \frac{1-\Omega_o^2}{1+\Omega_o^2} = \cos\omega_o$$

IIR Digital Filter Design Using Bilinear Transformation

- Example - Design a 2nd-order digital notch filter operating at a sampling rate of 400 Hz with a notch frequency at 60 Hz, 3-dB notch bandwidth of 6 Hz
- Thus $\omega_o = 2\pi(60/400) = 0.3\pi$

$$B_w = 2\pi(6/400) = 0.03\pi$$

- From the above values we get

$$\alpha = 0.90993$$

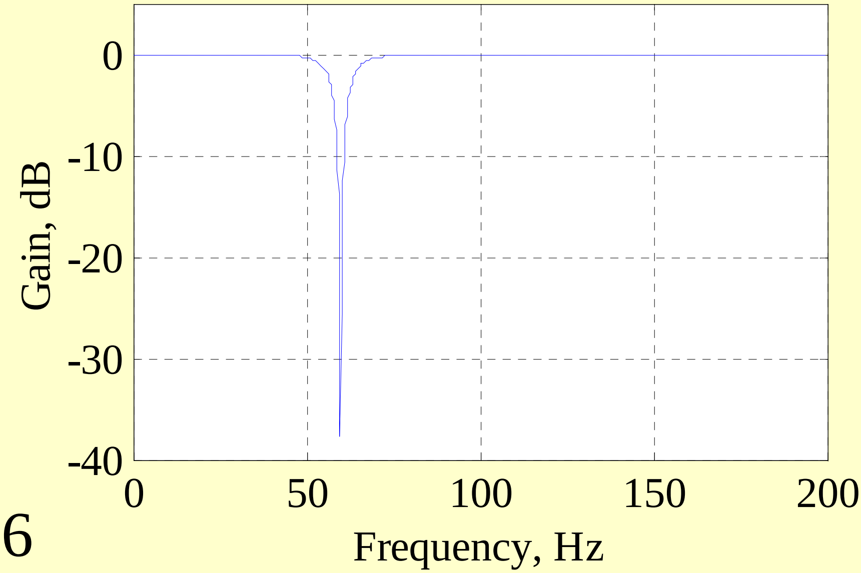
$$\beta = 0.587785$$

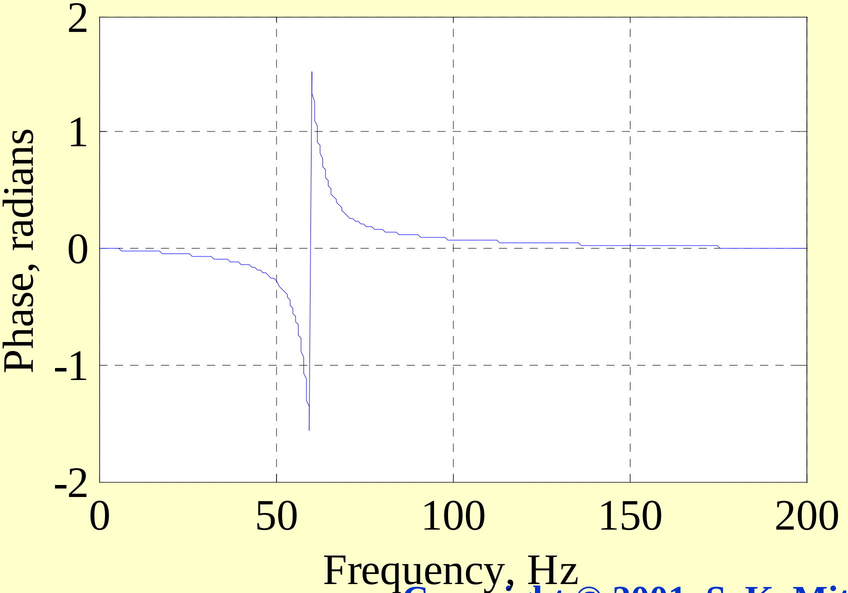
IIR Digital Filter Design Using Bilinear Transformation

- Thus

$$G(z) = \frac{0.954965 - 1.1226287\,z^{-1} + 0.954965\,z^{-2}}{1 - 1.1226287\,z^{-1} + 0.90993\,z^{-2}}$$

- The gain and phase responses are shown below





IIR Lowpass Digital Filter Design Using Bilinear Transformation

- Example - Design a lowpass Butterworth digital filter with $\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$, $\alpha_p \le 0.5\,\text{dB}$, and $\alpha_s \ge 15\ \text{dB}$
- Thus

$$\varepsilon^2 = 0.1220185 \qquad A^2 = 31.622777$$

- If $|G(e^{j0})| = 1$ this implies

$$20\log_{10}\left|G(e^{j0.25\pi})\right| \ge -0.5$$

$$20\log_{10}\left|G(e^{j0.55\pi})\right| \le -15$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

- Prewarping we get

$$\Omega_p = \tan(\omega_p / 2) = \tan(0.25\pi / 2) = 0.4142136$$

$$\Omega_s = \tan(\omega_s / 2) = \tan(0.55\pi / 2) = 1.1708496$$

- The inverse transition ratio is

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.8266809$$

- The inverse discrimination ratio is

$$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 15.841979$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

- Thus $$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 2.6586997$$

- We choose $N = 3$
- To determine Ω_c we use

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

- We then get

$$\Omega_c = 1.419915(\Omega_p) = 0.588148$$

- 3rd-order lowpass Butterworth transfer function for $\Omega_c = 1$ is

$$H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

- Denormalizing to get $\Omega_c = 0.588148$ we arrive at

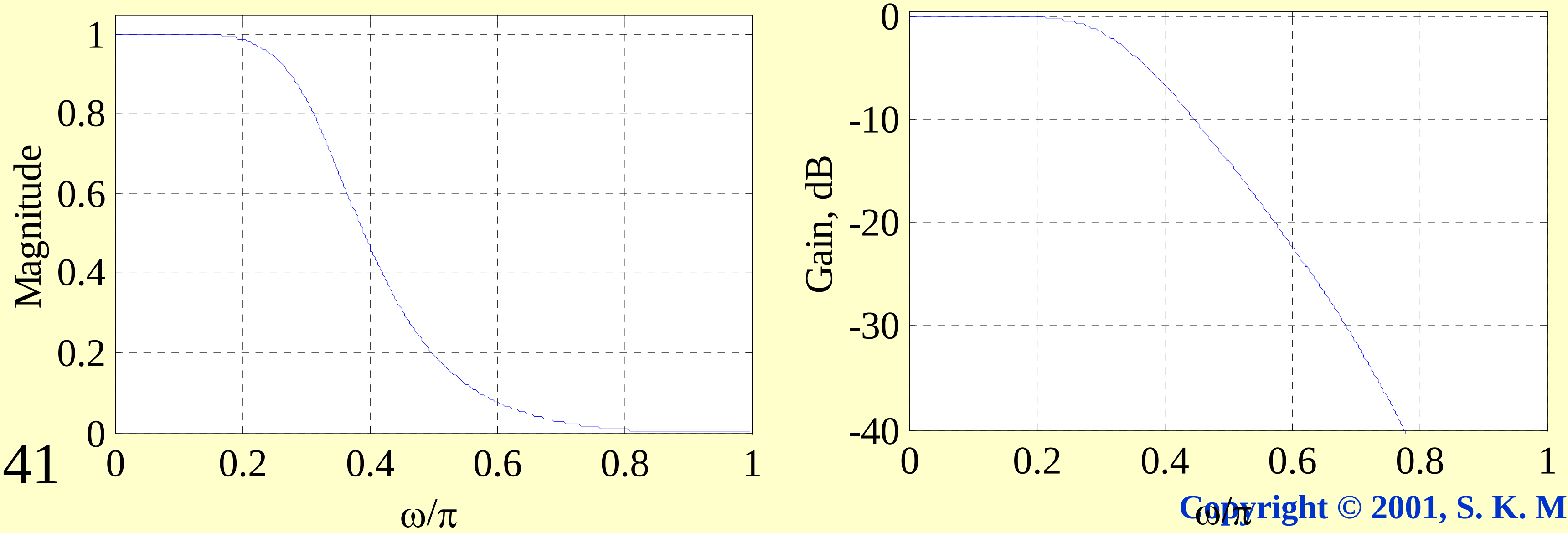
$$H_a(s) = H_{an}\left(\frac{s}{0.588148}\right)$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

- Applying bilinear transformation to $H_a(s)$ we get the desired digital transfer function

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

- Magnitude and gain responses of $G(z)$ shown below:



Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

- First Approach -

(1) Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type

(2) Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$

(3) Design analog lowpass filter $H_{LP}(s)$

Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

(4) Convert $H_{LP}(s)$ into $H_D(s)$ using inverse frequency transformation used in Step 2

(5) Design desired digital filter $G_D(z)$ by applying bilinear transformation to $H_{LP}(s)$

Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

- Second Approach -

  (1) Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type

  (2) Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$

Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

(3) Design analog lowpass filter $H_{LP}(s)$

(4) Convert $H_{LP}(s)$ into an IIR digital transfer function $G_{LP}(z)$ using bilinear transformation

(5) Transform $G_{LP}(z)$ into the desired digital transfer function $G_D(z)$

- We illustrate the first approach

IIR Highpass Digital Filter Design

- Design of a Type 1 Chebyshev IIR digital highpass filter
- Specifications: $F_p = 700$ Hz, $F_s = 500$ Hz, $\alpha_p = 1$ dB, $\alpha_s = 32$ dB, $F_T = 2$ kHz
- Normalized angular bandedge frequencies

$$\omega_p = \frac{2\pi F_p}{F_T} = \frac{2\pi \times 700}{2000} = 0.7\pi$$

$$\omega_s = \frac{2\pi F_s}{F_T} = \frac{2\pi \times 500}{2000} = 0.5\pi$$

IIR Highpass Digital Filter Design

- Prewarping these frequencies we get

$$\hat{\Omega}_p = \tan(\omega_p/2) = 1.9626105$$

$$\hat{\Omega}_s = \tan(\omega_s/2) = 1.0$$

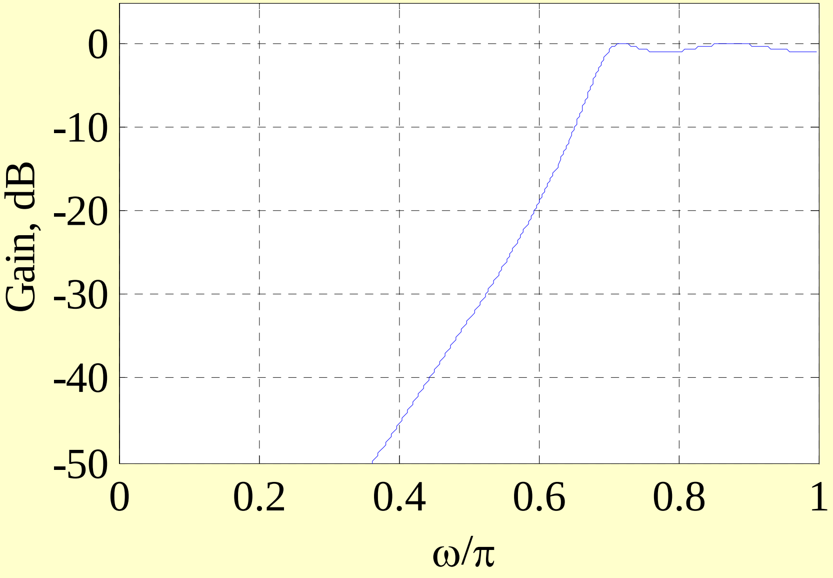
- For the prototype analog lowpass filter choose $\Omega_p = 1$
- Using $\Omega = -\dfrac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$ we get $\Omega_s = 1.962105$
- Analog lowpass filter specifications: $\Omega_p = 1$,

$\Omega_s = 1.926105$, $\alpha_p = 1$dB, $\alpha_s = 32$ dB

IIR Highpass Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = cheb1ord(1, 1.9626105, 1, 32, ’s’)
[B, A] = cheby1(N, 1, Wn, ’s’);
[BT, AT] = lp2hp(B, A, 1.9626105);
[num, den] = bilinear(BT, AT, 0.5);
```



IIR Bandpass Digital Filter Design

- Design of a Butterworth IIR digital bandpass filter
- Specifications: $\omega_{p1} = 0.45\pi$, $\omega_{p2} = 0.65\pi$,

$\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.75\pi$, $\alpha_p = 1$ dB, $\alpha_s = 40$ dB

- Prewarping we get

$$\hat{\Omega}_{p1} = \tan(\omega_{p1}/2) = 0.8540807$$

$$\hat{\Omega}_{p2} = \tan(\omega_{p2}/2) = 1.6318517$$

$$\hat{\Omega}_{s1} = \tan(\omega_{s1}/2) = 0.5095254$$

$$\hat{\Omega}_{s2} = \tan(\omega_{s2}/2) = 2.41421356$$

IIR Bandpass Digital Filter Design

- Width of passband $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.777771$

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = 1.393733$$

$$\hat{\Omega}_{s1}\hat{\Omega}_{s2} = 1.23010325 \neq \hat{\Omega}_o^2$$

- We therefore modify $\hat{\Omega}_{s1}$ so that $\hat{\Omega}_{s1}$and $\hat{\Omega}_{s2}$ exhibit geometric symmetry with respect to $\hat{\Omega}_o^2$
- We set $\hat{\Omega}_{s1} = 0.5773031$
- For the prototype analog lowpass filter we choose $\Omega_p = 1$

IIR Bandpass Digital Filter Design

- Using $\Omega = -\Omega_p \dfrac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega}\, B_w}$ we get

$$\Omega_s = \frac{1.393733 - 0.3332788}{0.5773031 \times 0.777771} = 2.3617627$$

- Specifications of prototype analog Butterworth lowpass filter:

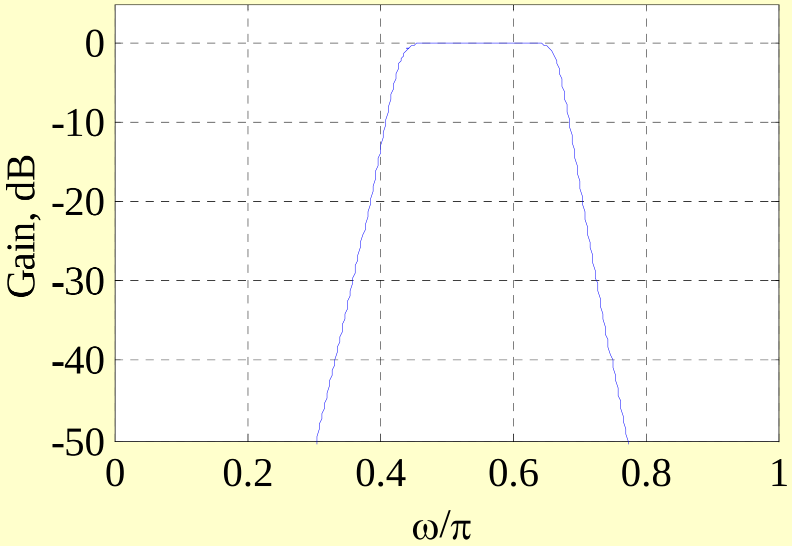
$\Omega_p = 1$, $\Omega_s = 2.3617627$, $\alpha_p = 1$ dB,

$\alpha_s = 40$ dB

IIR Bandpass Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = buttord(1, 2.3617627, 1, 40, ’s’)
[B, A] = butter(N, Wn, ’s’);
[BT, AT] = lp2bp(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```



IIR Bandstop Digital Filter Design

- Design of an elliptic IIR digital bandstop filter
- Specifications: $\omega_{s1} = 0.45\pi$, $\omega_{s2} = 0.65\pi$,

$\omega_{p1} = 0.3\pi$, $\omega_{p2} = 0.75\pi$, $\alpha_p = 1$ dB, $\alpha_s = 40$ dB

- Prewarping we get

$$\hat{\Omega}_{s1} = 0.8540806, \quad \hat{\Omega}_{s2} = 1.6318517,$$

$$\hat{\Omega}_{p1} = 0.5095254, \quad \hat{\Omega}_{p2} = 2.4142136$$

- Width of stopband $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 0.777771$

$$\hat{\Omega}_o^2 = \hat{\Omega}_{s2}\hat{\Omega}_{s1} = 1.393733$$

$$\hat{\Omega}_{p2}\hat{\Omega}_{p1} = 1.230103 \neq \hat{\Omega}_o^2$$

IIR Bandstop Digital Filter Design

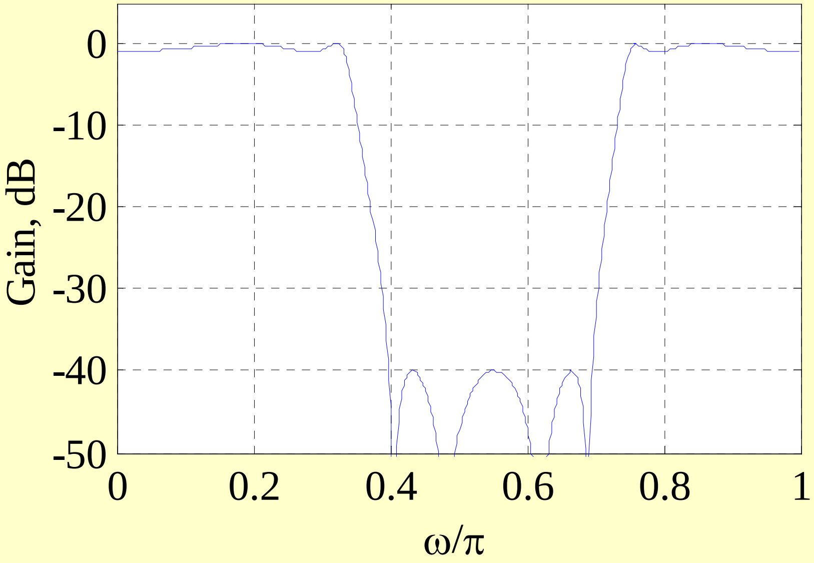
- We therefore modify $\hat{\Omega}_{p1}$ so that $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ exhibit geometric symmetry with respect to $\hat{\Omega}_o^2$
- We set $\hat{\Omega}_{p1} = 0.577303$
- For the prototype analog lowpass filter we choose $\Omega_s = 1$
- Using $\Omega = \Omega_s \dfrac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$ we get

$$\Omega_p = \frac{0.5095254 \times 0.777771}{1.393733 - 0.3332787} = 0.4234126$$

IIR Bandstop Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = ellipord(0.4234126, 1, 1, 40, ’s’);
[B, A] = ellip(N, 1, 40, Wn, ’s’);
[BT, AT] = lp2bs(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```


Gain, dB
ω/π