

Appendix K: Solution of State Equations for $t_0 \neq 0$

In Section 4.11 we used the state-transition matrix to perform a transformation taking $\mathbf{x}(t)$ from an initial time, $t_0 = 0$, to any time, $t \geq 0$, as defined in Eq. (4.109). What if we wanted to take $\mathbf{x}(t)$ from a different initial time, $t_0 \neq 0$, to any time $t \geq t_0$; would Eq. (4.109) and the state-transition matrix change? To find out, we need to convert Eq. (4.109) into a form that shows $t_0 \neq 0$ as the initial state rather than $t_0 = 0$ (Kuo, 1991).

Using Eq. (4.109), we find $\mathbf{x}(t)$ at t_0 to be

$$\mathbf{x}(t_0) = \Phi(t_0)\mathbf{x}(0) + \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.1})$$

Solving for $\mathbf{x}(0)$ by premultiplying both sides of Eq. (K.1) by $\Phi^{-1}(t_0)$ and rearranging,

$$\mathbf{x}(0) = \Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.2})$$

Substituting Eq. (K.2) into Eq. (4.109) yields

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)(\Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau) \\ &\quad + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \Phi(t)\Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi(t)\Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &\quad + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \quad (\text{K.3})$$

Since $\Phi(t) = e^{At}$ and $\Phi(-t) = e^{-At}$, $\Phi(t)\Phi(-t) = \mathbf{I}$. Hence,

$$\Phi^{-1}(t) = \Phi(-t) \quad (\text{K.4})$$

Therefore

$$\Phi(t)\Phi^{-1}(t_0) = e^{At}e^{-At_0} = e^{A(t-t_0)} = \Phi(t-t_0) \quad (\text{K.5})$$

Substituting Eq. (K.5) into Eq. (K.3) yields

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t-t_0)\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t-t_0)\Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &\quad + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \quad (\text{K.6})$$

But

$$\Phi(t - t_0)\Phi(t_0 - \tau) = e^{\mathbf{A}(t-t_0)}e^{\mathbf{A}(t_0-\tau)} = e^{\mathbf{A}(t-\tau)} = \Phi(t - \tau) \quad (\text{K.7})$$

Substituting Eq. (K.7) into Eq. (K.6),

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.8})$$

Combining the two integrals finally yields

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.9})$$

Equation (K.9) is more general than Eq. (4.109) in that it allows us to find $\mathbf{x}(t)$ after an initial time other than $t_0 = 0$. We can see that the state-transition matrix, $\Phi(t - t_0)$, is of a more general form than previously described. In particular, the state-transition matrix is also a function of the initial time. We conclude this section by deriving some important properties of $\Phi(t - t_0)$.

Using Eq. (K.4), the inverse of $\Phi(t - t_0)$ is

$$\Phi^{-1}(t - t_0) = \Phi(t_0 - t) \quad (\text{K.10})$$

Also, from Eq. (K.7),

$$\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\Phi(t_1 - t_0) \quad (\text{K.11})$$

which states that the transformation from t_0 to t_2 is the product of the transformation from t_0 to t_1 and the transformation from t_1 to t_2 .

Bibliography

Kuo, B. *Automatic Control Systems*, 6th ed. Prentice-Hall, Englewood Cliffs, NJ, 1991.

Copyright © 2015 John Wiley & Sons, Inc. All rights reserved.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Sections 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, Inc. 222 Rosewood Drive, Danvers, MA 01923, website www.copyright.com. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030-5774, (201)748-6011, fax (201)748-6008, website <http://www.wiley.com/go/permissions>.

Founded in 1807, John Wiley & Sons, Inc. has been a valued source of knowledge and understanding for more than 200 years, helping people around the world meet their needs and fulfill their aspirations. Our company is built on a foundation of principles that include responsibility to the communities we serve and where we live and work. In 2008, we launched a Corporate Citizenship Initiative, a global effort to address the environmental, social, economic, and ethical challenges we face in our business. Among the issues we are addressing are carbon impact, paper specifications and procurement, ethical conduct within our business and among our vendors, and community and charitable support. For more information, please visit our website: www.wiley.com/go/citizenship.

The software programs and experiments available with this book have been included for their instructional value. They have been tested with care but are not guaranteed for any particular purpose. The publisher and author do not offer any warranties or restrictions, nor do they accept any liabilities with respect to the programs and experiments.

AMTRAK is a registered trademark of National Railroad Passenger Corporation. **Adobe** and **Acrobat** are trademarks of Adobe Systems, Inc. which may be registered in some jurisdictions. **FANUC** is a registered trademark of FANUC, Ltd. **Microsoft**, **Visual Basic**, and **PowerPoint** are registered trademarks of Microsoft Corporation. **QuickBasic** is a trademark of Microsoft Corporation. **MATLAB** and **SIMULINK** are registered trademarks of The MathWorks, Inc. **The Control System Toolbox**, **LTI Viewer**, **Root Locus Design GUI**, **Symbolic Math Toolbox**, **Simulink Control Design**, and **MathWorks** are trademarks of The MathWorks, Inc. **LabVIEW** is a registered trademark of National Instruments Corporation. **Segway** is a registered trademark of Segway, Inc. in the United States and/or other countries. **Chevrolet Volt** is a trademark of General Motors LLC. Virtual plant simulations pictured and referred to herein are trademarks or registered trademarks of Quanser Inc. and/or its affiliates. © 2010 Quanser Inc. All rights reserved. Quanser virtual plant simulations pictured and referred to herein may be subject to change without notice. **ASIMO** is a registered trademark of Honda.

Evaluation copies are provided to qualified academics and professionals for review purposes only, for use in their courses during the next academic year. These copies are licensed and may not be sold or transferred to a third party. Upon completion of the review period, please return the evaluation copy to Wiley. Return instructions and a free of charge return shipping label are available at www.wiley.com/go/returnlabel. Outside of the United States, please contact your local representative.

Library of Congress Cataloging-in-Publication Data

Nise, Norman S.

Control systems engineering / Norman S. Nise, California State Polytechnic University, Pomona. — Seventh edition.
1 online resource.

Includes bibliographical references and index.

Description based on print version record and CIP data provided by publisher; resource not viewed.

ISBN 978-1-118-80082-9 (pdf) — ISBN 978-1-118-17051-9 (cloth : alk. paper)

1. Automatic control—Textbooks. 2. Systems engineering—Textbooks. I. Title.

TJ213

629.8—dc23

2014037468

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1