Appendix K: Solution of State Equations for $t_0 \neq 0$

In Section 4.11 we used the state-transition matrix to perform a transformation taking $\mathbf{x}(t)$ from an initial time, $t_0 = 0$, to any time, $t \geq 0$, as defined in Eq. (4.109). What if we wanted to take $\mathbf{x}(t)$ from a different initial time, $t_0 \neq 0$, to any time $t \geq t_0$; would Eq. (4.109) and the state-transition matrix change? To find out, we need to convert Eq. (4.109) into a form that shows $t_0 \neq 0$ as the initial state rather than $t_0 = 0$ (*Kuo, 1991*).

Using Eq. (4.109), we find $\mathbf{x}(t)$ at t_0 to be

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{x}(0) + \int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{Bu}(\tau)d\tau \tag{K.1}$$

Solving for $\mathbf{x}(0)$ by premultiplying both sides of Eq. (K.1) by $\mathbf{\Phi}^{-1}(t_0)$ and rearranging,

$$\mathbf{x}(0) = \mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{Bu}(\tau)d\tau \tag{K.2}$$

Substituting Eq. (K.2) into Eq. (4.109) yields

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t)(\mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{Bu}(\tau)d\tau \\ &\quad + \int_0^{t_0} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau \\ &= \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{Bu}(\tau)d\tau \\ &\quad + \int_0^{t} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau\end{aligned} \tag{K.3}$$

Since $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ and $\mathbf{\Phi}(-t) = e^{-\mathbf{A}t}$, $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$. Hence,

$$\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t) \tag{K.4}$$

Therefore

$$\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0} = e^{\mathbf{A}(t-t_0)} = \mathbf{\Phi}(t - t_0) \tag{K.5}$$

Substituting Eq. (K.5) into Eq. (K.3) yields

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \mathbf{\Phi}(t - t_0)\mathbf{\Phi}(t_0 - \tau)\mathbf{Bu}(\tau)d\tau \\ &\quad + \int_0^{t} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau\end{aligned} \tag{K.6}$$

But

$$\mathbf{\Phi}(t - t_0)\mathbf{\Phi}(t_0 - \tau) = e^{\mathbf{A}(t-t_0)}e^{\mathbf{A}(t_0-\tau)} = e^{\mathbf{A}(t_0-\tau)} = \mathbf{\Phi}(t - \tau) \tag{K.7}$$

Substituting Eq. (K.7) into Eq. (K.6),

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau + \int_0^{t} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau \tag{K.8}$$

Combining the two integrals finally yields

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - \tau)\mathbf{Bu}(\tau)d\tau \tag{K.9}$$

Equation (K.9) is more general than Eq. (4.109) in that it allows us to find $\mathbf{x}(t)$ after an initial time other than $t_0 = 0$. We can see that the state-transition matrix, $\mathbf{\Phi}(t - t_0)$, is of a more general form than previously described. In particular, the state-transition matrix is also a function of the initial time. We conclude this section by deriving some important properties of $\mathbf{\Phi}(t - t_0)$.

Using Eq. (K.4), the inverse of $\mathbf{\Phi}(t - t_0)$ is

$$\mathbf{\Phi}^{-1}(t - t_0) = \mathbf{\Phi}(t_0 - t) \tag{K.10}$$

Also, from Eq. (K.7),

$$\mathbf{\Phi}(t_2 - t_0) = \mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) \tag{K.11}$$

which states that the transformation from t_0 to t_2 is the product of the transformation from t_0 to t_1 and the transformation from t_1 to t_2.

Bibliography

Kuo, B. *Automatic Control Systems*, 6th ed. Prentice-Hall, Englewood Cliffs, NJ, 1991.

Founded in 1807, John Wiley & Sons, Inc. has been a valued source of knowledge and understanding for more than 200 years, helping people around the world meet their needs and fulfill their aspirations. Our company is built on a foundation of principles that include responsibility to the communities we serve and where we live and work. In 2008, we launched a Corporate Citizenship Initiative, a global effort to address the environmental, social, economic, and ethical challenges we face in our business. Among the issues we are addressing are carbon impact, paper specifications and procurement, ethical conduct within our business and among our vendors, and community and charitable support. For more information, please visit our website: www.wiley.com/go/citizenship.

Library of Congress Cataloging-in-Publication Data

Nise, Norman S.
Control systems engineering / Norman S. Nise, California State Polytechnic University, Pomona. — Seventh edition.
1 online resource.
Includes bibliographical references and index.
Description based on print version record and CIP data provided by publisher; resource not viewed.
ISBN 978-1-118-80082-9 (pdf) — ISBN 978-1-118-17051-9 (cloth : alk. paper)
1. Automatic control–Textbooks. 2. Systems engineering–Textbooks. I. Title.
TJ213
629.8–dc23

2014037468

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1