

Appendix M: Root Locus Rules: Derivations

M.1 Derivation of the Behavior of the Root Locus at Infinity (Kuo, 1987)

Let the open-loop transfer function be represented as follows:

$$KG(s)H(s) = \frac{K(s^m + a_1s^{m-1} + \dots + a_m)}{(s^{m+n} + b_1s^{m+n-1} + \dots + b_{m+n})} \quad (\text{M.1})$$

or

$$KG(s)H(s) = \frac{K}{\left(\frac{s^{m+n} + b_1s^{m+n-1} + \dots + b_{m+n}}{s^m + a_1s^{m-1} + \dots + a_m} \right)} \quad (\text{M.2})$$

Performing the indicated division in the denominator, we obtain

$$KG(s)H(s) = \frac{K}{s^n + (b_1 - a_1)s^{n-1} + \dots} \quad (\text{M.3})$$

In order for a pole of the closed-loop transfer function to exist,

$$KG(s)H(s) = -1 \quad (\text{M.4})$$

Assuming large values of s that would exist as the locus moves toward infinity, Eq. (M.3) becomes

$$s^n + (b_1 - a_1)s^{n-1} = -K \quad (\text{M.5})$$

Factoring out s^n , Eq. (M.5) becomes

$$s^n \left(1 + \frac{b_1 - a_1}{s} \right) = -K \quad (\text{M.6})$$

Taking the n th root of both sides, we have

$$s \left(1 + \frac{b_1 - a_1}{s} \right)^{1/n} = -K^{1/n} \quad (\text{M.7})$$

If the term

$$\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} \quad (\text{M.8})$$

is expanded into an infinite series where only the first two terms are significant,¹ we obtain

$$s \left(1 + \frac{b_1 - a_1}{ns}\right) = (-K)^{1/n} \quad (\text{M.9})$$

Distributing the factor S on the left-hand side yields

$$s + \frac{b_1 - a_1}{n} = (-K)^{1/n} \quad (\text{M.10})$$

Now, letting $s = \sigma + j\omega$ and $(-K)^{1/n} = |K|^{1/n} e^{j(2k+1)\pi/n}$, where

$$(-1)^{1/n} = e^{j(2k+1)\pi/n} = \cos\left(\frac{(2k+1)\pi}{n}\right) + j \sin\left(\frac{(2k+1)\pi}{n}\right) \quad (\text{M.11})$$

Eq. (M.10) becomes

$$\sigma + j\omega + \frac{b_1 - a_1}{n} = |K|^{1/n} \left[\cos\frac{(2k+1)\pi}{n} + j \sin\frac{(2k+1)\pi}{n} \right] \quad (\text{M.12})$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$. Setting the real and imaginary parts of both sides equal to each other, we obtain

$$\sigma + \frac{b_1 - a_1}{n} = |K|^{1/n} \cos\frac{(2k+1)\pi}{n} \quad (\text{M.13a})$$

$$\omega = |K|^{1/n} \sin\frac{(2k+1)\pi}{n} \quad (\text{M.13b})$$

Dividing the two equations to eliminate $|K|^{1/n}$, we obtain

$$\frac{\sigma + \frac{b_1 - a_1}{n}}{\omega} = \frac{\cos\frac{(2k+1)\pi}{n}}{\sin\frac{(2k+1)\pi}{n}} \quad (\text{M.14})$$

Finally, solving for ω , we find

$$\omega = \left[\tan\frac{(2k+1)\pi}{n} \right] \left[\sigma + \frac{b_1 - a_1}{n} \right] \quad (\text{M.15})$$

The form of this equation is that of a straight line,

$$\omega = M(\sigma - \sigma_0) \quad (\text{M.16})$$

where the slope of the line, M , is

$$M = \tan\frac{(2k+1)\pi}{n} \quad (\text{M.17})$$

¹This is a good approximation since s is approaching infinity for the region applicable to the derivation.

Thus, the angle of the line in radians with respect to the positive extension of the real axis is

$$\theta = \frac{(2k + 1)\pi}{n} \quad (\text{M.18})$$

and the σ intercept is

$$\sigma_0 = -\left[\frac{b_1 - a_1}{n} \right] \quad (\text{M.19})$$

From the theory of equations,²

$$b_1 = -\sum \text{finite poles} \quad (\text{M.20a})$$

$$a_1 = -\sum \text{finite zeros} \quad (\text{M.20b})$$

Also, from Eq. (M.1),

$$\begin{aligned} n &= \text{number of finite poles} - \text{number of finite zeros} \\ &= \#\text{finite poles} - \#\text{finite zeros} \end{aligned} \quad (\text{M.21})$$

By examining Eq. (M.16), we conclude that the root locus approaches a straight line as the locus approaches infinity. Further, this straight line intersects the σ axis at

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \quad (\text{M.22})$$

which is obtained by substituting Eqs. (M.20)

Let us summarize the results: *The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept and the angle with respect to the real axis as follows:*

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \quad (\text{M.23})$$

$$\theta = \frac{(2k + 1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \quad (\text{M.24})$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$. Notice that the running index, k , in Eq. (M.24) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

M.2 Derivation of Transition Method for Breakaway and Break-in Points

The *transition* method for finding real-breakaway and break-in points without differentiating can be derived by showing that the natural log of $1/[G(\sigma)H(\sigma)]$ has a zero derivative at the same value of σ as $1/[G(\sigma)H(\sigma)]$ (Franklin, 1991).

We now show that if we work with the natural log we can eliminate the step of differentiation.

² Given an n th-order polynomial of the form $s^n + a_{n-1}s^{n-1} + \dots$, the coefficient, a_{n-1} , is the negative sum of the roots of the polynomial.

First find the derivative of the natural log of $1/[G(\sigma)H(\sigma)]$ and set it equal to zero. Thus,

$$\frac{d}{d\sigma} \ln \left[\frac{1}{G(\sigma)H(\sigma)} \right] = G(\sigma)H(\sigma) \frac{d}{d\sigma} \left[\frac{1}{G(\sigma)H(\sigma)} \right] = 0 \quad (\text{M.25})$$

Since $G(\sigma)H(\sigma)$ is not zero at the breakaway or break-in points, letting

$$\frac{d}{d\sigma} \ln \left[\frac{1}{G(\sigma)H(\sigma)} \right] = 0 \quad (\text{M.26})$$

will thus yield the same value of σ as letting

$$\frac{d}{d\sigma} \left[\frac{1}{G(\sigma)H(\sigma)} \right] = 0 \quad (\text{M.27})$$

Hence,

$$\begin{aligned} \frac{d}{d\sigma} \ln \left[\frac{1}{G(\sigma)H(\sigma)} \right] &= \frac{d}{d\sigma} \ln \left[\frac{(\sigma + p_1)(\sigma + p_2) \cdots (\sigma + p_n)}{(\sigma + z_1)(\sigma + z_2) \cdots (\sigma + z_m)} \right] \\ &= \frac{d}{d\sigma} [\ln(\sigma + p_1) + \ln(\sigma + p_2) \cdots \ln(\sigma + p_n) \\ &\quad - \ln(\sigma + z_1) - \ln(\sigma + z_2) \cdots - \ln(\sigma + z_m)] \\ &= \frac{1}{\sigma + p_1} + \frac{1}{\sigma + p_2} \cdots + \frac{1}{\sigma + p_n} - \frac{1}{\sigma + z_1} - \frac{1}{\sigma + z_2} \cdots \\ &\quad - \frac{1}{\sigma + z_m} = 0 \end{aligned} \quad (\text{M.28})$$

or

$$\sum_{i=1}^n \frac{1}{\sigma + p_i} = \sum_{i=1}^m \frac{1}{\sigma + z_i} \quad (\text{M.29})$$

where z_i and p_i are the negatives of the zero and pole values of $G(s)H(s)$, respectively. Equation (M.29) can be solved for σ , the real axis values that minimize or maximize K , yielding the breakaway and break-in points without differentiating.

Bibliography

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