Appendix M: Root Locus Rules: Derivations

M.1 Derivation of the Behavior of the Root Locus at Infinity (Kuo, 1987)

Let the open-loop transfer function be represented as follows:

$$KG(s)H(s) = \frac{K(s^m + a_1 s^{m-1} + \cdots + a_m)}{(s^{m+n} + b_1 s^{m+n-1} + \cdots + b_{m+n})} \tag{M.1}$$

or

$$KG(s)H(s) = \frac{K}{\left(\dfrac{s^{m+n} + b_1 s^{m+n-1} + \cdots + b_{m+n}}{s^m + a_1 s^{m-1} + \cdots + a_m}\right)} \tag{M.2}$$

Performing the indicated division in the denominator, we obtain

$$KG(s)H(s) = \frac{K}{s^n + (b_1 - a_1)s^{n-1} + \cdots} \tag{M.3}$$

In order for a pole of the closed-loop transfer function to exist,

$$KG(s)H(s) = -1 \tag{M.4}$$

Assuming large values of s that would exist as the locus moves toward infinity, Eq. (M.3) becomes

$$s^n + (b_1 - a_1)s^{n-1} = -K \tag{M.5}$$

Factoring out s^n, Eq. (M.5) becomes

$$s^n\left(1 + \frac{b_1 - a_1}{s}\right) = -K \tag{M.6}$$

Taking the nth root of both sides, we have

$$s\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} = -K^{1/n} \tag{M.7}$$

If the term

$$\left(1+\frac{b_1-a_1}{s}\right)^{1/n} \tag{M.8}$$

is expanded into an infinite series where only the first two terms are significant,[1] we obtain

$$s\left(1+\frac{b_1-a_1}{ns}\right)=(-K)^{1/n} \tag{M.9}$$

Distributing the factor S on the left-hand side yields

$$s+\frac{b_1-a_1}{n}=(-K)^{1/n} \tag{M.10}$$

Now, letting $s=\sigma+j\omega$ and $(-K)^{1/n}=|K^{1/n}|e^{j(2k+1)\pi/n}$, where

$$(-1)^{1/n}=e^{j(2k+1)\pi/n}=\cos\left(\frac{(2k+1)\pi}{n}\right)+j\sin\left(\frac{(2k+1)\pi}{n}\right) \tag{M.11}$$

Eq. (M.10) becomes

$$\sigma+j\omega+\frac{b_1-a_1}{n}=\left|K^{1/n}\right|\left[\cos\frac{(2k+1)\pi}{n}+j\sin\frac{(2k+1)\pi}{n}\right] \tag{M.12}$$

where $k=0,\pm 1,\pm 2,\pm 3,\ldots$ Setting the real and imaginary parts of both sides equal to each other, we obtain

$$\sigma+\frac{b_1-a_1}{n}=\left|K^{1/n}\right|\cos\frac{(2k+1)\pi}{n} \tag{M.13a}$$

$$\omega=|k^{1/n}|\sin-\frac{(2k+1)\pi}{n} \tag{M.13b}$$

Dividing the two equations to eliminate $|K^{1/n}|$, we obtain

$$\frac{\sigma+\dfrac{b_1-a_1}{n}}{\omega}=\frac{\cos\dfrac{(2k+1)\pi}{n}}{\sin\dfrac{(2k+1)\pi}{n}} \tag{M.14}$$

Finally, solving for ω, we find

$$\omega=\left[\tan\frac{(2k+1)\pi}{n}\right]\left[\sigma+\frac{b_1-a_1}{n}\right] \tag{M.15}$$

The form of this equation is that of a straight line,

$$\omega=M(\sigma-\sigma_0) \tag{M.16}$$

where the slope of the line, M, is

$$M=\tan\frac{(2k+1)\pi}{n} \tag{M.17}$$

[1] This is a good approximation since s is approaching infinity for the region applicable to the derivation.

Thus, the angle of the line in radians with respect to the positive extension of the real axis is

$$\theta = \frac{(2k+1)\pi}{n} \tag{M.18}$$

and the σ intercept is

$$\sigma_0 = -\left[\frac{b_1 - a_1}{n}\right] \tag{M.19}$$

From the theory of equations,[2]

$$b_1 = -\sum \text{finite poles} \tag{M.20a}$$

$$a_1 = -\sum \text{finite zeros} \tag{M.20b}$$

Also, from Eq. (M.1),

$$\begin{aligned} n &= \text{number of finite poles} - \text{number of finite zeros} \\ &= \#\text{finite poles} - \#\text{finite zeros} \end{aligned} \tag{M.21}$$

By examining Eq. (M.16), we conclude that the root locus approaches a straight line as the locus approaches infinity. Further, this straight line intersects the σ axis at

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.22}$$

which is obtained by substituting Eqs. (M.20)

Let us summarize the results: *The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept and the angle with respect to the real axis as follows:*

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.23}$$

$$\theta = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.24}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ Notice that the running index, k, in Eq. (M.24) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

M.2 Derivation of Transition Method for Breakaway and Break-in Points

The *transition* method for finding real-breakaway and break-in points without differentiating can be derived by showing that the natural log of $1/[G(\sigma)H(\sigma)]$ has a zero derivative at the same value of σ as $1/[G(\sigma)H(\sigma)]$ (*Franklin, 1991*).

We now show that if we work with the natural log we can eliminate the step of differentiation.

[2] Given an nth-order polynomial of the form $s^n + a_{n-1}s^{n-1} + \cdots$, the coefficient, a_{n-1}, is the negative sum of the roots of the polynomial.

First find the derivative of the natural log of $1/[G(\sigma)H(\sigma)]$ and set it equal to zero. Thus,

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = G(\sigma)H(\sigma)\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.25}$$

Since $G(\sigma)H(\sigma)$ is not zero at the breakaway or break-in points, letting

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.26}$$

will thus yield the same value of σ as letting

$$\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.27}$$

Hence,

$$\begin{aligned}
\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] &= \frac{d}{d\sigma}\ln\left[\frac{(\sigma+p_1)(\sigma+p_2)\cdots(\sigma+p_n)}{(\sigma+z_1)(\sigma+z_2)\cdots(\sigma+z_m)}\right] \\
&= \frac{d}{d\sigma}\left[\ln(\sigma+p_1)+\ln(\sigma+p_2)\cdots\ln(\sigma+p_n)\right. \\
&\quad \left. -\ln(\sigma+z_1)-\ln(\sigma+z_2)\cdots-\ln(\sigma+z_m)\right] \\
&= \frac{1}{\sigma+p_1}+\frac{1}{\sigma+p_2}\cdots+\frac{1}{\sigma+p_n}-\frac{1}{\sigma+z_1}-\frac{1}{\sigma+z_2}\cdots \\
&\quad -\frac{1}{\sigma+z_m} = 0
\end{aligned} \tag{M.28}$$

or

$$\sum_{i=1}^{n}\frac{1}{\sigma+p_i} = \sum_{i=1}^{m}\frac{1}{\sigma+z_i} \tag{M.29}$$

where z_i and p_i are the negatives of the zero and pole values of $G(s)H(s)$, respectively. Equation (M.29) can be solved for σ, the real axis values that minimize or maximize K, yielding the breakaway and break-in points without differentiating.

Bibliography

Franklin, G. F.; Powell, J. D.; and Emami-Naeini, A. *Feedback Control of Dynamic Systems*, 2d ed. Addison-Wesley, Reading, Mass., 1991.

Kuo, B. *Automatic Control Systems*, 5th ed. Prentice-Hall, Englewood Cliffs, N.J., 1987.

Founded in 1807, John Wiley & Sons, Inc. has been a valued source of knowledge and understanding for more than 200 years, helping people around the world meet their needs and fulfill their aspirations. Our company is built on a foundation of principles that include responsibility to the communities we serve and where we live and work. In 2008, we launched a Corporate Citizenship Initiative, a global effort to address the environmental, social, economic, and ethical challenges we face in our business. Among the issues we are addressing are carbon impact, paper specifications and procurement, ethical conduct within our business and among our vendors, and community and charitable support. For more information, please visit our website: www.wiley.com/go/citizenship.

The software programs and experiments available with this book have been included for their instructional value. They have been tested with care but are not guaranteed for any particular purpose. The publisher and author do not offer any warranties or restrictions, nor do they accept any liabilities with respect to the programs and experiments.

Evaluation copies are provided to qualified academics and professionals for review purposes only, for use in their courses during the next academic year. These copies are licensed and may not be sold or transferred to a third party. Upon completion of the review period, please return the evaluation copy to Wiley. Return instructions and a free of charge return shipping label are available at www.wiley.com/go/returnlabel. Outside of the United States, please contact your local representative.

Library of Congress Cataloging-in-Publication Data

Nise, Norman S.
Control systems engineering / Norman S. Nise, California State Polytechnic University, Pomona. — Seventh edition.
1 online resource.
Includes bibliographical references and index.
Description based on print version record and CIP data provided by publisher; resource not viewed.
ISBN 978-1-118-80082-9 (pdf) — ISBN 978-1-118-17051-9 (cloth : alk. paper)
1. Automatic control–Textbooks. 2. Systems engineering–Textbooks. I. Title.
TJ213
629.8–dc23

2014037468

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1