STEADY-STATE ERRORS

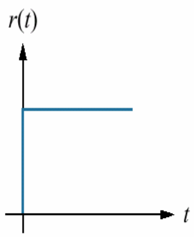
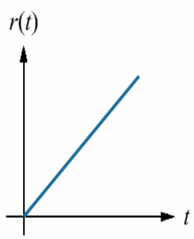
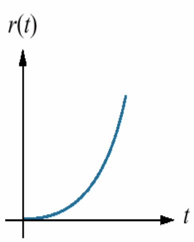
In this lecture you will learn the following :

- ❖How to find the steady-state error for a unity feedback system
- ❖How to specify a system's steady-state error performance
- ❖How to find steady-state error for a nonunity feedback system
- ❖How to find the steady-state error for systems represented in state-space

In lecture 1, we saw that control systems analysis and design focus on three specificaitons: (1) transient response, (2) stability, (3) steady state errors.

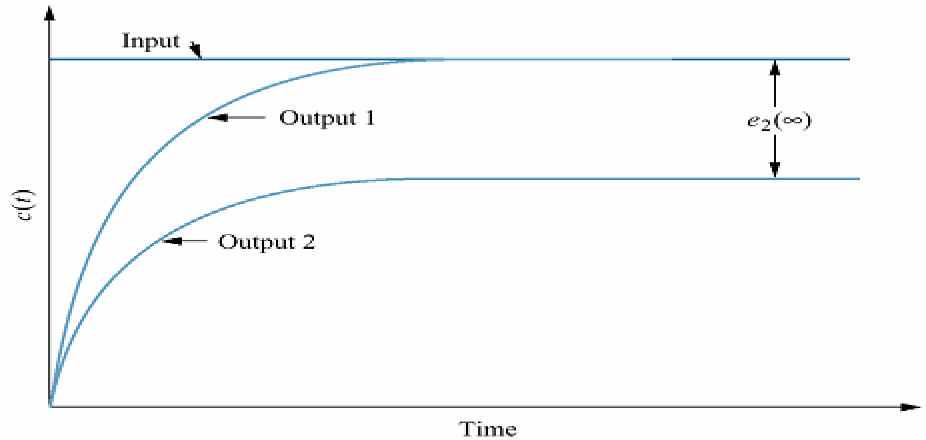
We have talked about transient response and stability by now. Now we are ready to examine the steady state errors.

Steady-state error is the difference between the input and output for a prescribed test input as t goes to infinity. The test input used for steady-state error analysis and design are summarized in the following table.

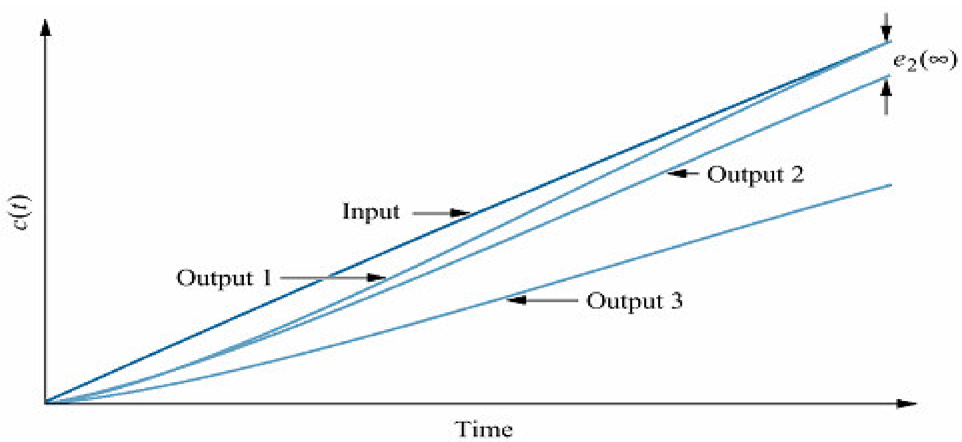
Waveform	**Name**	**Physical interpretation**	**Time function**	**Laplace transform**
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Since we are concerned with the difference between the input and output of a feedback control system after steady-state has been reached, our discussion is limited to stable systems. Thus the engineer must check the system for stability while performing steady-state error analysis and design. However, in order to focus on the topic, we assume that all systems in examples and problems in this lecture are stable. For practice you may want to test some of the systems for stability.

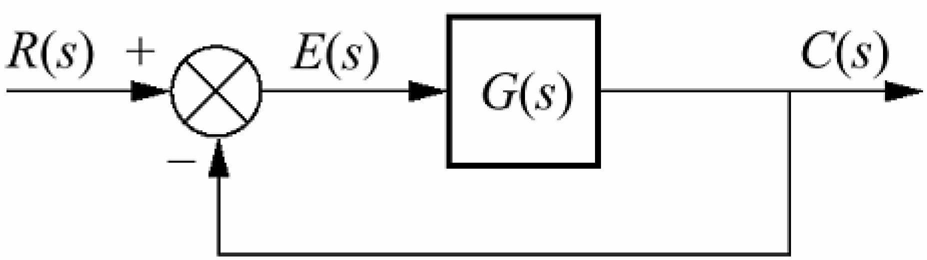
Evaluating Steady-State Errors : Let us examine the concept of staedy-state error.



In this figure, a step input and two possible outputs are shown. Output 1 has zero steady-state error, and output 2 has a finite steady-state error. A similar example which has a ramp input is shown in the following figure.



Output 1 has a zero steady-state error. Output 2 has a finite steady-state error. Output 3 has a infinite steady state error as time goes to infinity. Let us now look at the error from the perspective of the most general block diagram.



Since the error is the difference between the input and the output of a system, we assume a closed loop system, the error E(s) is the difference between the output C(s) and the input R(s) for a unity feedback system as shown in the figure.

STEADY STATE ERROR FOR UNITY FEEDBACK SYSTEMS

Steady-state error can be calculated from a system's closed loop transfer function, T(s), or the open loop transfer function, G(s), for unity feedback systems as shown in figure.

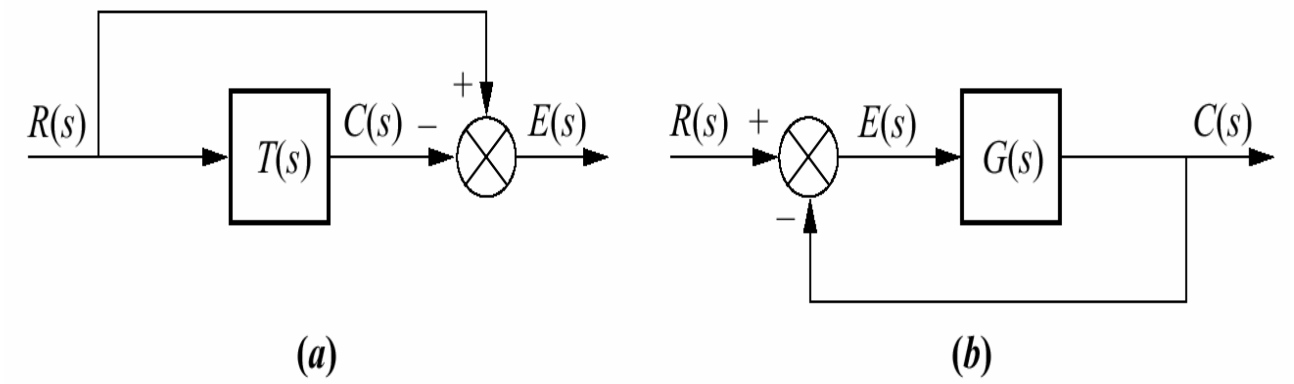


Figure 7.3
Closed-loop control system error:
a. general representation;
b. representation for unity feedback systems

We begin by deriving the system's steady-state error in terms of the closed loop transfer function, T(s), in order to introduce the subject and definitions. Next we obtain insight into the factors affecting steady-state error by using the open loop transfer function, G8S), in unity feedback systems for our calculations. Later in this lecture we generalize this discussion to nonunity feedback systems.

Steady-State Error in Terms of T(s) : Consider the figure(a). To find E(s), we write

$$E(s)=R(s) - C(s)\ , \qquad \text{but} \qquad C(s) = R(s)T(s).$$

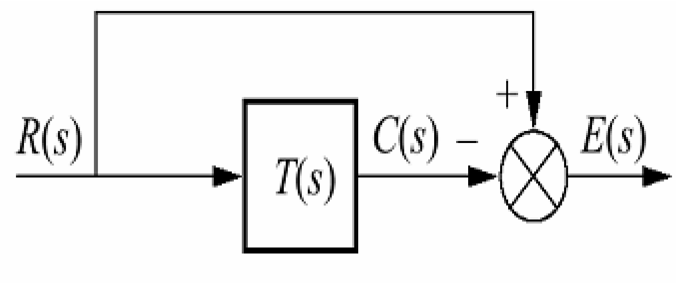
Substituting C(s) into R(s) and solving for E(s) yields $E(s) = R(s)[1-T(s)]$. Although the last equation allows us to solve for e(t) at any time t, we are interested in the final value of the error, e(∞). Appliying the final value theorem, we obtain

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

Substituting the last equation E(s) = R(s)[1 - T(s)] to the last equation we obtained yields

$$e(\infty) = \lim_{s\to 0} sR(s)[1 - T(s)]$$

Example : Find the steay-state error for the system of the following figure if T(s)=5/(s²+7s+10) and the input is a unit step.

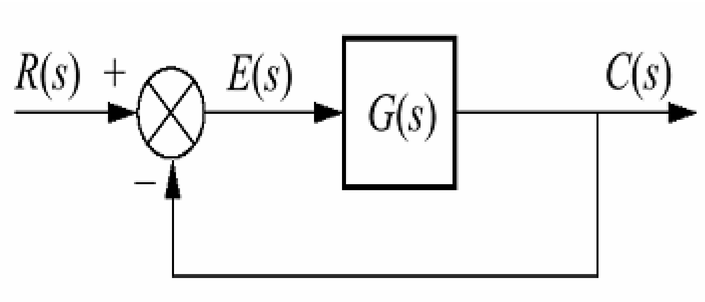


Solution : R(s)=1/s , T(s)=5/(s2+7s+10) , E(s)=R(s)[1-T(s)] → $E(s) = \dfrac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$

Since T(s) is stable and, subsequently, E(s) does not have right half plane poles or jω poles other than at the origin, we can apply the final value theorem.

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \quad \Rightarrow \quad e(\infty) = 1/2$$

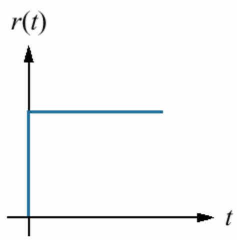
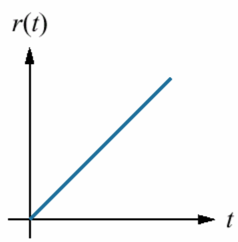
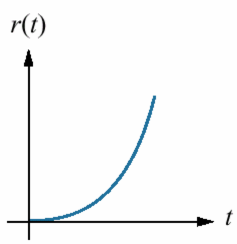
Steady-state Errors in Terms of G(s) : Consider the feedback control system shown in the figure. E(s) = R(s) – C(s) and C(s) = E(s)G(s) → $E(s) = \dfrac{R(s)}{1 + G(s)}$



Applying the final value theorem yields

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

The three test signals we use to establish specification for a control system steady-state error characteristics are shown in the following table. Let us take each input and evaluate its effect on the steady-state error by using the equation $e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Step input : R(s)=1/s $\rightarrow$ we find $e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$

The term $\lim_{s\to 0} G(s)$ is the dc gain of the forward transfer function. In order to have zero steady-state error, we must get $\lim_{s\to 0} G(s) = \infty$. To get this, G(s) must take on the following form :

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots\ldots\ldots}{s^n(s+p_1)(s+p_2)\ldots\ldots\ldots\ldots}$$

and for the limit to be infinite, the denominator must be equal to zero as *s* goes to zero. Thus n≥1; that is, at least one pole must be at the origin. Since division by s in the frequency domain, we are also saying that at least one pure integration must be present in the forward path. If there are no integrations, then n=0, thse steady state error is finite.

Ramp input : R(s)=1/s^2 , $e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$

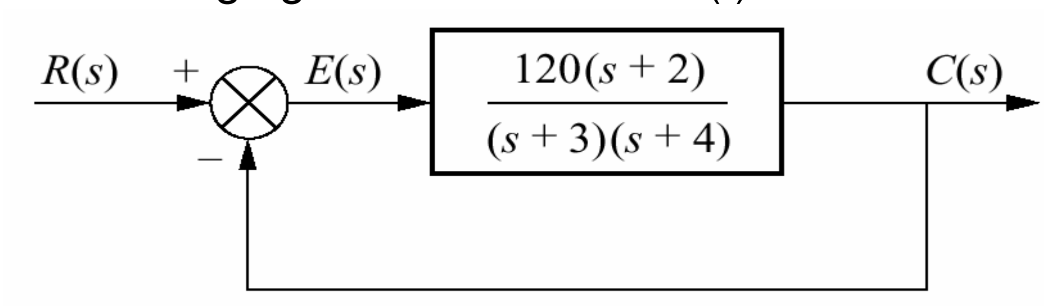
To have zero steady-state error, we must have $\lim_{s\to 0} sG(s) = \infty$. To satisfy this, G(s) must take the same form mentioned above, except that n≥2. In other words, there must be at least two integrations in the forward path. If one integration exists, steady-state error will be finite. If there are no integrations, then steady-state error will be infinite.

Parabolic input : R(s)=1/s³ , $e(\infty) = e_{parabola}(\infty) = \lim_{s \to 0} \dfrac{s(1/s^3)}{1+G(s)} = \lim_{s \to 0} \dfrac{1}{s^2 + s^2 G(s)} = \dfrac{1}{\lim_{s \to 0} s^2 G(s)}$

In order to have zero steady-state error, we must have $\lim_{s \to 0} s^2 G(s) = \infty$

To satisfy this, n≥3. In other words, there must be at least three integrations in the forward path. If two integrations exist, steady-state error will be finite. If there is only one or less integration, steady-state error will be infinite.

Example : Find the steady-state errors for inputs of 5u(t), 5tu(t), and 5t²u(t) to the system shown in the following figure. The function u(t) is the unit step.



Solution :

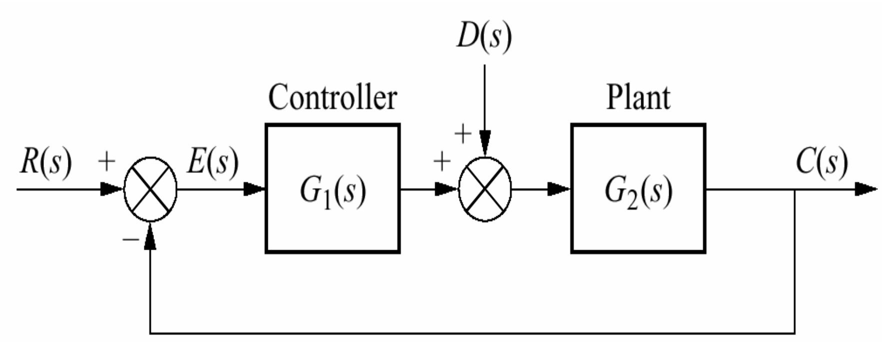
$$e(\infty) = e_{step}(\infty) = \frac{5}{1 + \lim_{s \to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s \to 0} sG(s)} = \frac{5}{0} = \infty$$

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s \to 0} s^2 G(s)} = \frac{10}{0} = \infty$$

STEADY-STATE ERROR FOR DISTURBANCES

Feedback control systems are used to compansate for disturbance or unwanted inputs enter a system The following figure shows a feedback control system with a disturbance, D(s), injected between the controller and the plant. We now re-derive the expression for steady-state error with the disturbance included.



The transform of output is given by

$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$

Substituting $C(s) = R(s) – E(s)$ into the output equation and solving for $E(s)$ yields

$$E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(S)G_2(s)}D(s)$$

where we can think of $1/[1+G_1(s)G_2(s)]$ as a transfer function relating E(s) to R(s) and $-G_2(s)/[1+G_1(s)G_2(s)]$ as a transfer function relating E(s) to D(s). To find steady-state value of the error, we apply the final value theorem to equation of E(s) and obtain

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}R(s) - \lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)$$

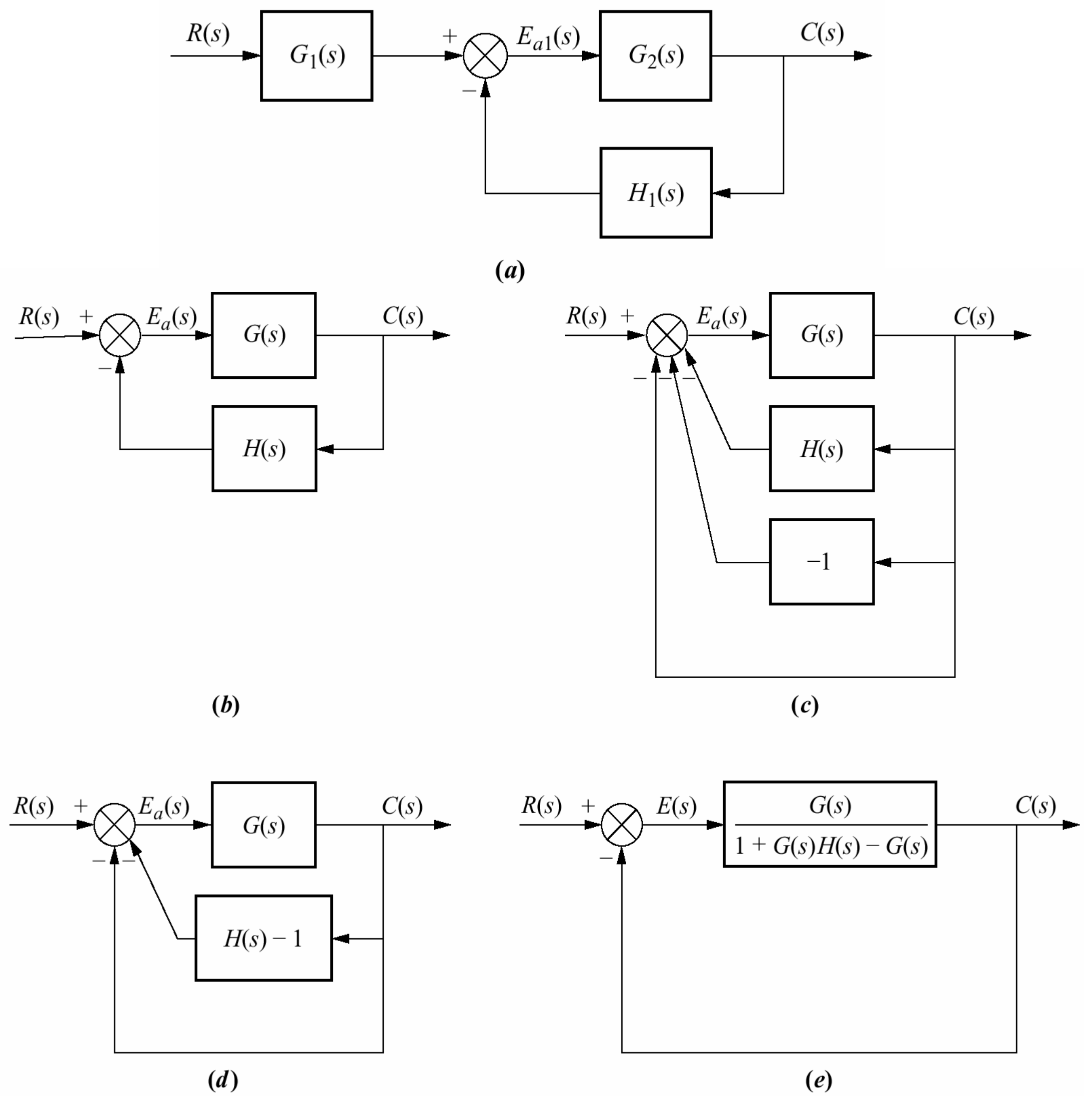
where

$$e_R(\infty) = \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}R(s) \quad \text{and} \quad e_D(s) = -\lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)$$

The first term $e_R(\infty)$ is the steady-state error due to R(s), which we have already obtained. The second term $e_D(\infty)$ is the steady-state error due to disturbance. Let us explore the conditions on $e_D(\infty)$ that must exist to reduce the error due to disturbance.

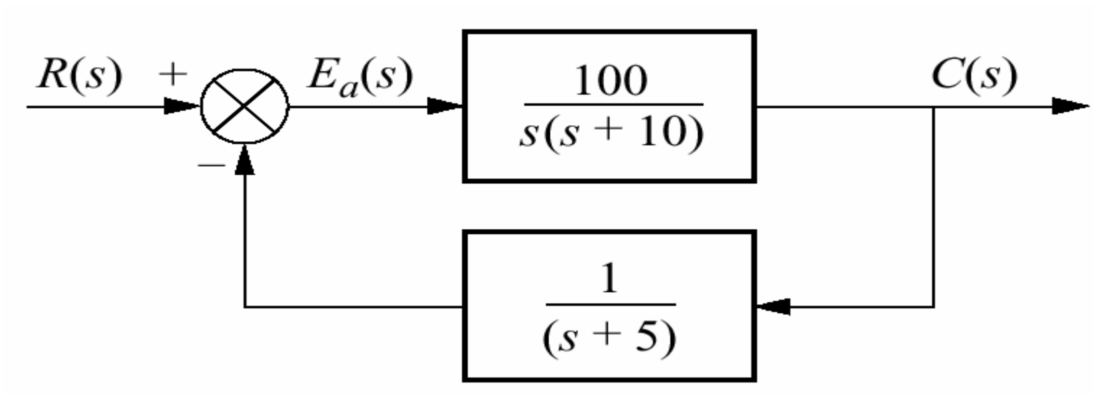
STEADY-STATE ERROR FOR NONUNITY FEEDBACK SYSTEMS

Control system often do not have unity feedback because of the compensation used to improve performance or because of the physical model foe the system. A general feedback system, showing the input transducer, $G_1(s)$, controller and plant, $G_2(s)$, and feedback, $H_1(s)$, is shown in the following figure.



Pushing the input transducer, $G_1(s)$, controller and plant, $G_2(s)$, and feedback, $H_1(s)$, is shown in figure(b), where $G(s)=G_1(s)G_2(s)$ and $H(s)=H_1(s)/G_1(s)$. To convert a nonunity feedback system to a unity feedback system, form a unity feedback system by adding and subtracting unity feedback paths, as shown in figure(c). This step requires the input and the output units be the same. Next combine H(s) with the negative unity feedback as shown in figure(d). Finally combine the feedback system consisting of G(s) and [H(s) – 1] as shown in figure(e). Notice that the final figure shows E(s)=R(s)-C(s) explicitly and we can use all algorithms axplained before.

Example : For the system shown in the figure, find the steady state error for a unit step input.



Solution : Note that $G(s) = \dfrac{100}{s(s+10)}$ and $\mathrm{H(s)} = \dfrac{1}{\mathrm{s}+5}$

Using the equivalent unity feedback system block diagram shown in figure(e) in the previous page, the equivalent forward path transfer function $G_e(s)$ is calculated as

$$G_e(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{100}{s^3+15s^2-50s-400}$$

We can calculate the steady-state error anymore :

$$e(\infty) = \frac{1}{1+\lim_{s\to 0} G_e(s)} = \frac{1}{1+\left(\dfrac{100\times 5}{-400}\right)} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step.

SENSITIVITY

During the design process the engineer may want to consider the extent to which changes in system parameters affect the behavior of a system. Ideally, parameter changes due to heat or other causes should not appreciably affect a system's performance. The degree to which changes in system parameter affect system transfer functions, and hence the performance, is called *sensitivity*. A system wtih zero sensitivity (that is, changes in the system parameter have no effect on the transfer function) is ideal. For example, assume the function *F = K/(K+a).* If *K*=10 and *a*=100, then *F*=0.091. If parameter *a* triples to 300, then *F*=0.032. We see that a fractional change in parameter *a* of (300-100)/100=2 (%200 change), yields a change in the function *F* of (0.032-0.091)/0.091=-0.65 (- %65 change). Thus the function *F* has reduced sensitivity to changes in parameter *a*. As we proceed, we will see that another advantage of feedback is that in general it affords reduced sensistivity to parameter changes.

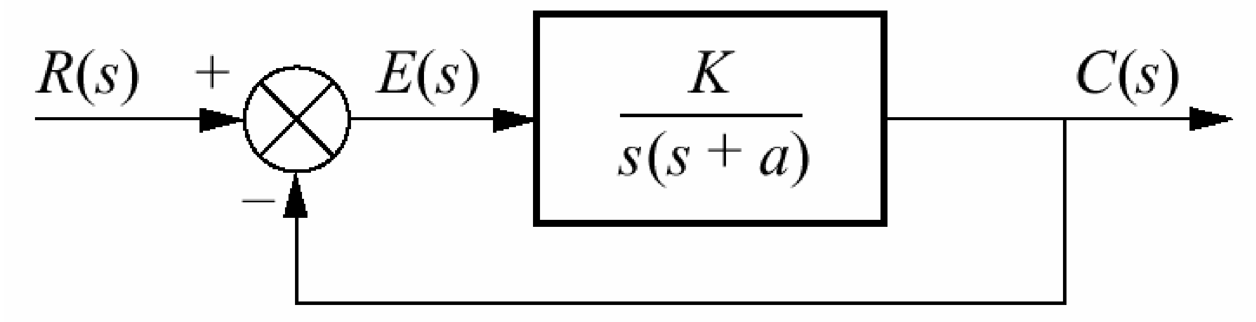
Based upon the this discussion, let us formalize a definition of sensistivity : Sensitivity is the ratio of the fractional change in the function to the fractional change of the parameter as the fractional change of parameter approaches zero. That is,

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{Fractional \text{ change in the function, F}}{Fractional \text{ change in the parameter, P}} = \lim_{\Delta \mathrm{P} \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{\mathrm{F}} \frac{\Delta \mathrm{F}}{\Delta \mathrm{P}}$$

which reduces to

$$S_{F:P} = \frac{P}{\mathrm{F}} \frac{\delta \mathrm{F}}{\delta \mathrm{P}}$$

Example : Given the system in the figure, calculate the sensistivity of the closed loop transfer function to changes in the parameter *a*. How would you reduce the the sensistivity.



Solution : The closed loop transfer function is $T(s) = \dfrac{K}{s^2 + as + K}$

The sensitivity is given by

$$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\left(\dfrac{K}{s^2 + as + K}\right)}\left(\frac{-Ks}{\left(s^2 + as + K\right)^2}\right) = \frac{-as}{s^2 + as + K}$$

which is, in part, a function of the value of s. For any value of *s*, however, an increase in *K* reduces the sensitivity of the closed loop transfer function to changes in the parameter *a*.

STEADY-STATE ERROR FOR THE THE SYSTEM IN STATE SPACE

Up to this point we have evaluated the steady-state error for systems modeled as transfer functions. We will now discuss how to evaluate the steady-state error for systems represented in state space.

A single input, single output system represented in state space can be analyzed for steady-state error using the final value theorem and the closed loop transfer function. Consider the closed loop system represented in state space :

$$\dot{x} = Ax + Br$$
$$y = Cx$$

The Laplace transform of the error is E(s) = R(s) – Y(s) . The output equation is Y(s)=R(s)T(s) where T(s) is the closed loop transfer function. Inserting Y(s) to E(s) yields E(s) = R(s) [1-T(s)] . We know from the lecture 3 that T(s) = C(sI-A)$^{-1}$B+D . Inserting this to E(s) yields E(s) = R(s)[1-C(sI-A)$^{-1}$B]. Applying the final value theorem, we have

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)[1 - C(sI - A)^{-1}B]$$

Example : Evaluate the steady-state error for the system described by the following equation for unit stepand unit ramp inputs. Use the final value theorem.

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}; \qquad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \qquad \mathrm{C} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

Solution : Using the formulation we evaluated before,

$$e(\infty) = \lim_{s \to 0} sR(s)\left(1 - \frac{s+4}{s^3 + 6s^2 + 13s + 20}\right)$$

$$= \lim_{s \to 0} sR(s)\left(\frac{s^3 + 6s^2 + 12s + 16}{s^3 + 6s^2 + 13s + 20}\right)$$

For a unit step, R(s)=1/s, and e(∞)=4/5. For a unit ramp, R(s) = $1/s^2$ and e(∞)=∞