Lecture 7

Why modulate or why use a carrier wave? Why not couple a baseband signal to the antenna directly?

Ex: $f = 3000$ Hz

$C = 3 \cdot 10^8$ m/s

$$\lambda = \frac{c}{f} = \frac{3 \cdot 10^8}{3 \cdot 10^3} = 1 \cdot 10^5 \text{ m} =$$

$\lambda/4$ typical standard for mobile phones

size of antenna 25 km!

If instead $f = 1800$ MHz

$$\frac{\lambda}{4} = \frac{1}{4} \frac{3 \cdot 10^8}{18 \cdot 10^8} = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24} \text{ m}$$

$$= \underline{\underline{4 \text{ cm}}}$$ somewhat more suitable

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Band pass modulation

Multiplication by a carrier wave; a sinusoid

characterized by three features:

Amplitude, phase, and frequency

$$s(t) = A(t) \cos \theta(t) = A(t) \cos[\omega t + \phi(t)]$$

We distinguish between coherent and non-coherent detection where coherent means that the receiver exploits the carrier's phase to detect the signals.

needs to know the exact phase of carrier	Does not need exact phase
Coherent	Non-coherents
PSK, FSK, ASK, CPM	DPSK, FSK, ASK, CPM
Hybrids	Hybrids.

(requires a phase lock to the incoming signal)

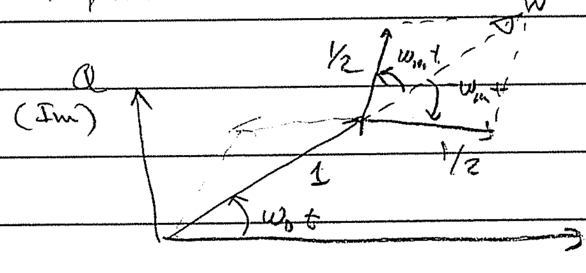
Modulation examples

(AM) Let $\cos w_m t$ modulate a carrier $e^{jw_0 t}$ so that

$$s(t) = \mathrm{Re}\left(e^{jw_0 t}\left(1 + \cos w_m t\right)\right)$$

$$= \mathrm{Re}\left(e^{jw_0 t}\left(1 + \frac{e^{jw_m t}}{2} + \frac{e^{-jw_m t}}{2}\right)\right)$$

physical signal



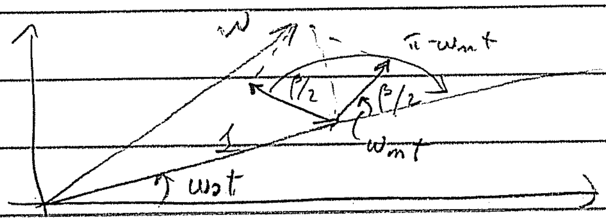
when the phasor rotates and w_m varies the resulting vector W will vary in magnitude i.e. the amplitude varies.

FM (narrow band FM)

Let $\beta \cos \omega_m t$ modulate the carrier $e^{j\omega_0 t}$ so that the transmitted sequence is

$$s(t) = \mathrm{Re}\left\{ e^{j\omega_0 t} \left(1 - (\beta \cos \omega_m t) \right) \right\}$$

$$= \mathrm{Re}\left\{ e^{j\omega_0 t} \left(1 - \frac{\beta}{2} e^{-j\omega_m t} + \frac{\beta}{2} e^{j\omega_m t} \right) \right.$$



When phasor rotates an $\omega_m t$ is varying results in the phasor slowing down or speeding up = frequency changes.

PSK general expression

symbol energy

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[\omega_0 t + \phi_i(t)\right] \qquad 0 \le t \le T, \quad i = 1, \dots M$$

symbol duration

The phase term will have M discrete values

e.g. $\phi_i(t) = \dfrac{2\pi i}{M}$ $\quad i = 1, \dots, M$

000	$-\ 0$
001	$-\ \pi/4$
010	$-\ \pi/2$
011	$-\ 3\pi/4$
100	$-\ \pi$
$\vdots$	

BPSK: $\phi_i = 0, \pi$ $\qquad$ 8PSK: $\phi_i = 0, \pi/4, \pi/2, 3\pi/4, \pi \dots$

FSK General expression

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi) \qquad 0 \le t \le T, \quad i = 1, \dots M$$

ω_i has M discrete values (and ϕ is arbitrary constant.)
e.g. 2, 4, 8...

ASK general expression

$$S_i(t) = \sqrt{\frac{2E_i(t)}{T}}\cos(\omega_0 t + \phi) \qquad 0 \le t \le T$$
$$i = 1, \dots, M = 2^k$$

The amplitude term $\sqrt{\frac{2E_i(t)}{T}}$ will have M discrete values.
ϕ arbitrary constant.
Binary ASK = on-off keying

APK (amplitude phase keying)

General expression

$$S_i(t) = \sqrt{\frac{2E_i(t)}{T}}\cos(\omega_0 t + \phi_i(t)) \qquad 0 \le t \le T$$
$$i = 1, \dots M$$

Example $M = 8$, 4 $E_i(t)$ and 4 $\phi_i(t)$
If ordered in a rectangular constellation = QAM
(QAM = Quadrature Amplitude Modulation)

x

Q: Why $\sqrt{\frac{2E}{T}}$?
A: If $S(t) = A\cos\omega t$, then

$$\text{peak value } A = \sqrt{2}\,A_{rms} = \sqrt{2A_{rms}^2} = \sqrt{2\overset{\text{average power}}{\bar{P}}} = \sqrt{\frac{2\overset{\text{Average energy}}{\bar{E}}}{\underset{\text{symbol period}}{T}}}$$

Coherent Detection of PSK

ex BPSK (Antipodal case)

$$s_1(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi) \qquad 0 < t \le T$$

$$s_2(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi + \pi) = -\sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi)$$

e.g. $=0$

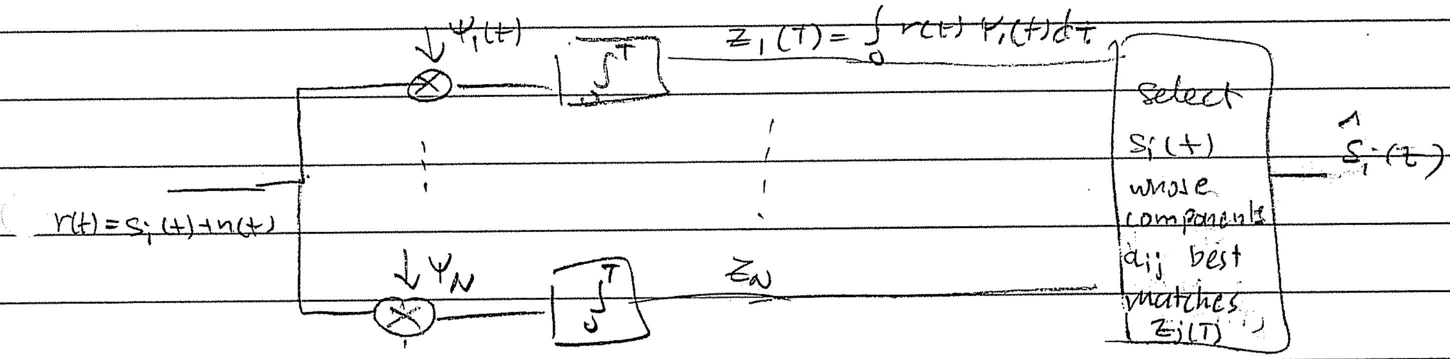
Now, let $\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_0 t$

Note: we only need one basis function in this case, N=1 below

Thus,

$$s_1 = \sqrt{E}\, \psi_1(t)$$
$$s_2 = -\sqrt{E}\, \psi_1(t)$$

$-\sqrt{E}$ $\sqrt{E}$ $\psi_1(t)$ In phase component



Calculate the expected value of z_i ~~belonging~~ to the class s_i

$$E\{z_1 \mid s_1\} = E\left\{ \int_0^T r(t)\psi_1(t)\,dt \mid s_1 \right\} = E\left\{ \int_0^T (s_i(t) + n(t))\psi_1(t)\,dt \right\}$$

$$= E\left\{ \int_0^T \Big(\sqrt{E}\, \underbrace{\psi_1(t)^2}_{\frac{2}{T}\cos^2 \omega_0 t} + n(t)\underbrace{\psi_1(t)}_{\sqrt{\frac{2}{T}}\cos\omega_0 t} \Big) dt \right\} = E\{\sqrt{E} \int_0^T \langle \psi_1, \psi_1 \rangle dt\} + \underbrace{\int_0^T E\, n(t)\psi}_{=0} = \sqrt{E}$$

$$E\{z_1 \mid s_2\} = E\left\{ \int_0^T -\sqrt{E}\,\psi_1(t)^2 + n(t)\psi_1(t)\,dt \right\} = \dots = -\sqrt{E}$$

$E\,n(t)\psi_1(t) = \psi_1(t) E n(t) = 0$ since $n(t)$ is assumed to have zero mean.

Choose the $s_1(t)$ if $z_1 > \gamma_0 = 0$ else choose s_2

Coherent detection of Multiple phase shift keying (MPSK)

transmitted signal

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\Big(\omega_0 t - \underbrace{\frac{2\pi i}{M}}_{\phi_i}\Big) \qquad 0 \le t \le T, \quad i = 1, \ldots, M$$

Assume 4ary PSK i.e. $M = 4 = 2^k$; $k = 2$

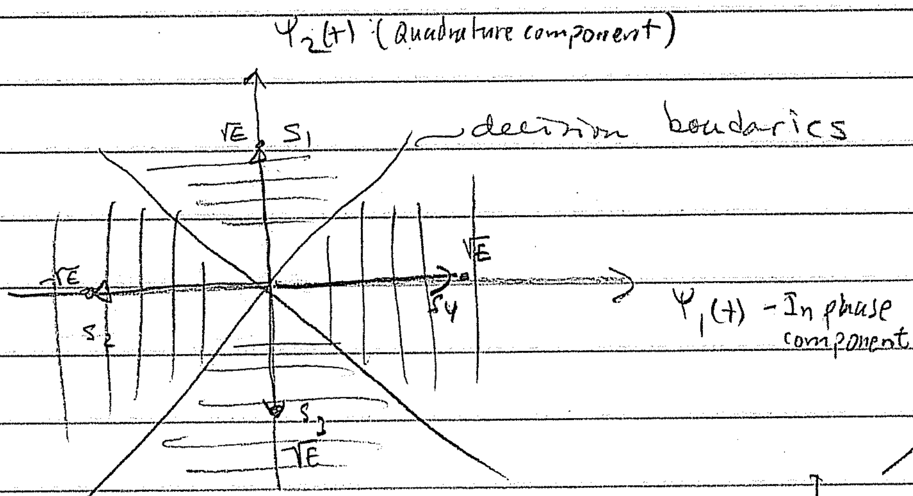
and assume orthogonal basis functions $(\psi_1(t) \perp \psi_2(t))$

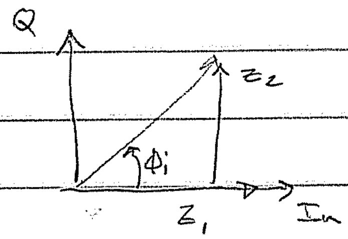
$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_0 t \qquad \psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega_0 t$$

[transmitted signal = linear combination of basis functions]

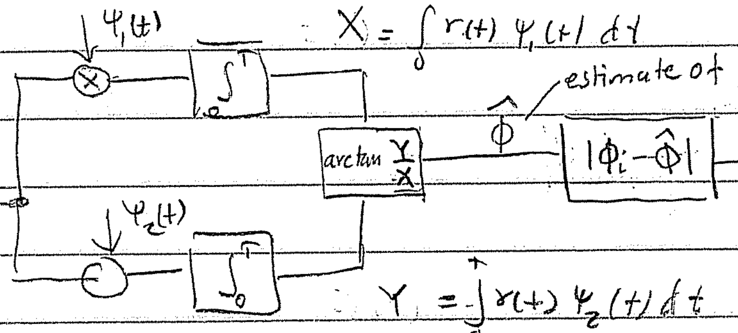
$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t) = \sqrt{E}\cos(\phi_i)\,\psi_1 + \sqrt{E}\sin(\phi_i)\,\psi_2(t)$$

$$\left[\sqrt{\frac{2}{T}}\cos(\omega_0 t - \phi_i) = \left[\sqrt{\frac{2}{T}}\cos(\omega_0 t)\cos(\phi_i) + \sqrt{\frac{2}{T}}\sin(\omega_0 t)\sin(\phi_i)\right] = \cancel{= \cos(\phi_i)\psi_1 + \sin(\phi_i)\psi_2}\right]$$





MPSK - Demodulator



$$r(t) = s_i(t) + n(t)$$

$$X = \int_0^T r(t)\,\psi_1(t)\,dt$$

$$\hat{\phi}$$

$$|\phi_i - \hat{\phi}|$$

$$Y = \int_0^T r(t)\,\psi_2(t)\,dt$$

Example MPSK

8 PSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_o t - \Phi_i) \qquad 0 < t \le T$$

$$= \sqrt{E}\cos\Phi_i \sqrt{\tfrac{2}{T}}\cos\omega_o t + \sqrt{E}\sin\Phi_i \sqrt{\tfrac{2}{T}}\sin\omega_o t$$

$i = 1, \ldots 8 = M = 2^3$

$\Phi_i = \frac{2\pi i}{M}$

Assume orthogonal basis functions

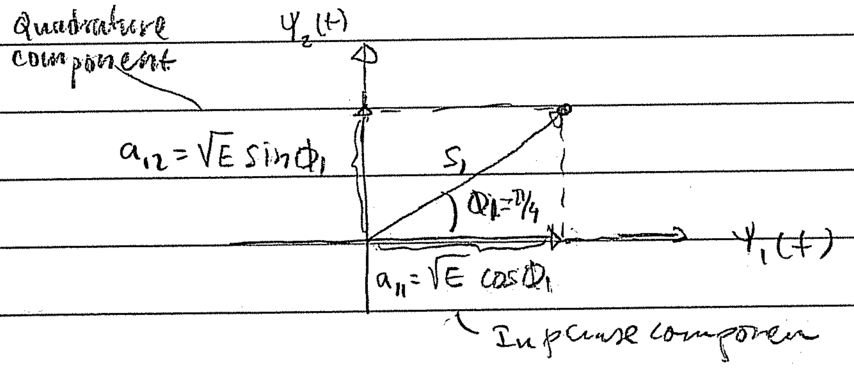
$$\psi_1(t) = \sqrt{\tfrac{2}{T}} \cos\omega_o t \qquad \psi_2(t) = \sqrt{\tfrac{2}{T}} \sin\omega_o t$$

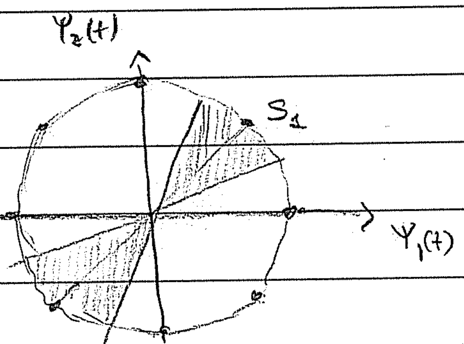
(normalization factor, can also be regarded as a rectangular pulse shaping)

Assume that $\Phi_1 = \frac{\pi}{4}$ is the information to be transmitted

The transmitted signal is then given by

$$s_1(t) = \underbrace{\sqrt{E}\cos\Phi_1}_{a_{11}} \psi_1(t) + \underbrace{\sqrt{E}\sin\Phi_1}_{a_{12}} \psi_2(t) = a_{11}\psi_1 + a_{12}\psi_2$$





Now, $s_1(t)$ is transmitted. Use the MPSK-demodulator to retrieve Φ_1

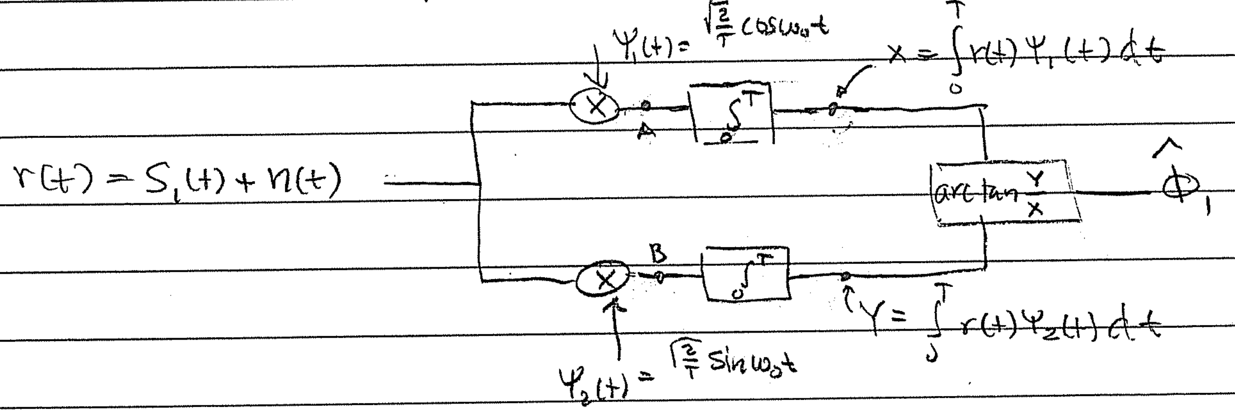


FIG 1

The mathematical expression given at point X and Y are

$$X = \int_0^T \left(a_{11}\varphi_1(t) + a_{12}\varphi_2(t) + n(t)\right)\varphi_1(t)\,dt$$

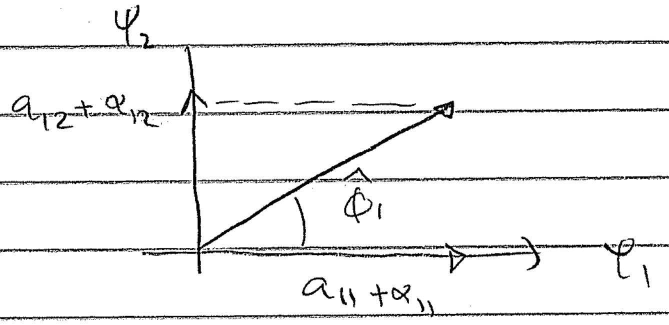
$$= \int_0^T a_{11}\varphi_1(t)\varphi_1(t)\,dt + \int_0^T a_{12}\varphi_2(t)\varphi_1(t)\,dt + \underbrace{\int_0^T n(t)\varphi_1(t)\,dt}_{\alpha_{11}}$$

$$= a_{11} + \alpha_{11}$$

$$Y = \int_0^T \left(a_{11}\varphi_1(t) + a_{12}\varphi_2(t) + n(t)\right)\varphi_2(t)\,dt$$

$$= \dots = a_{12} + \alpha_{12}$$

Note that $X = a_{11}$ + noise is the (noisy) in phase component and Y is the quadrature component of $s_1(t)$



$$\hat{\phi}_1 = \arctan\left(\frac{Y}{X}\right) = \arctan\left(\frac{a_{12}+\alpha_{12}}{a_{11}+\alpha_{11}}\right)$$

$$\underset{\phi}{=} \arctan\left(\frac{\sqrt{E}\sin\frac{\pi}{4}}{\sqrt{E}\cos\frac{\pi}{4}}\right) = \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \arctan(1)$$

if the noise is zero $\qquad = \pi/4$

if noise is not zero $\hat{\phi}_1 \approx \frac{\pi}{4}$

The decision circuit gives $\hat{\phi} = \phi_1 = \frac{\pi}{4}$ which was transmitted.

Note that

$$x = \int_0^T r(t)\,\psi_1(t)\,dt \quad \text{and} \quad y = \int_0^T r(t)\,\psi_2(t)\,dt \qquad (*)$$

are operating on passband signals.

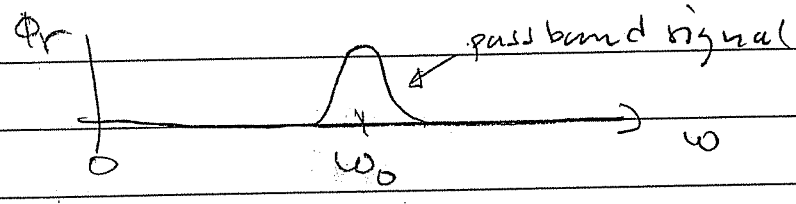
By expressing the signal in point A i.e. the integrand $r(t)\,\psi_1(t)$ we obtain

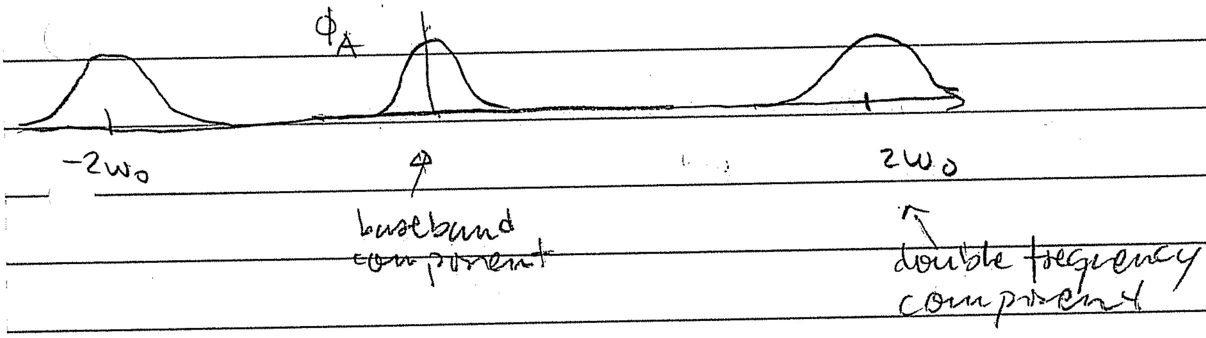
$$A = \left(a_{11}\psi_1(t) + a_{12}\psi_2(t)\right)\psi_1(t) + n(t)\,\psi_1(t)$$

$$= a_{11}\,\psi_1(t)^2 + a_{12}\,\psi_2(t)\,\psi_1(t) + n(t)\,\psi_1(t)$$

$$= a_{11}\left(\sqrt{\tfrac{2}{T}}\right)^2 (\cos\omega_0 t)^2 + a_{12}\left(\sqrt{\tfrac{2}{T}}\right)^2 \sin\omega_0 t\,\cos\omega_0 t + \underbrace{n(t)\,\psi_1(t)}_{\triangleq\,\tilde{n}(t)}$$

$$= \frac{2a_{11}}{T}\,\frac{1}{2}\left(1 + \cos 2\omega_0 t\right) + \frac{2a_{12}}{T}\,\frac{1}{2}\sin 2\omega_0 t + \tilde{n}(t)$$





Performing the integration of the signal A gives

$$X = a_{11}\frac{2}{T}\int_0^T \frac{1}{2}\left(1 + \cos 2\omega_0 t\right)dt + a_{12}\frac{2}{T}\int_0^T \frac{1}{2}\sin 2\omega_0 t\,dt + \int_0^T \tilde{n}(t)\,dt$$

$$= a_{11}\frac{2}{T}\left[\frac{t}{2}\right]_0^T + a_{11}\frac{2}{T}\left[\frac{\sin 2\omega_0 t}{4\omega_0}\right]_0^T + a_{12}\frac{2}{T}\left[-\frac{\cos 2\omega_0 t}{4\omega_0}\right]_0^T + \int_0^T \tilde{n}(t)\,dt$$

(Note the role of the normalization factor $\sqrt{\frac{2}{T}}$)

$$= a_{11}\frac{2}{T}\cdot\frac{T}{2} + a_{11}\frac{2}{T}\left[\frac{\sin 2\omega_0 T}{4\omega_0}\right] + a_{12}\frac{2}{T}\left[\frac{1}{4\omega_0} - \frac{\cos 2\omega_0 T}{4\omega_0}\right] + n_{11}$$

$$X = a_{11} + \underbrace{a_{11} \frac{\sin 2\omega_0 T}{2\omega_0 T}}_{\mathrm{I}} + \underbrace{a_{12}\left[\frac{1}{2\omega_0 T} - \frac{\cos 2\omega_0 T}{2\omega_0 T}\right]}_{\mathrm{II}} + n_{11}$$

Term I ≈ 0 since for $\omega_0 T \gg 1$ the numerator is bounded by 1 and the denominator is very large ($\omega_0 T \gg 1 \Leftrightarrow \omega_0 \gg \frac{1}{T} \Leftrightarrow$ carrier frequency $\gg$ symbol rate, or equivalently, the carrier frequency $\gg$ bandwidth of the baseband signal.

In Term II the first factor ≈ 0 since $\omega_0 T \gg 1$.
Similarly, for the $\frac{\cos 2\omega_0 T}{2\omega_0 T}$ term, it is ≈ 0 for $\omega_0 T \gg 1$.

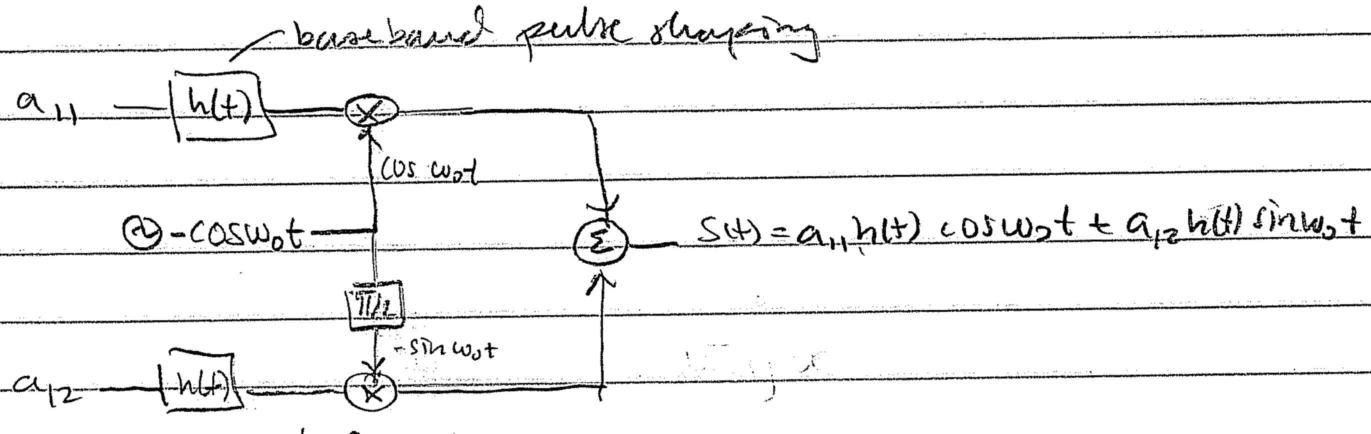
Thus, $X = a_{11} + n_{11}$

In the same way we obtain $Y = a_{12} + n_{12}$

Note that the integration effectively cancels out the $2\omega_0 T$ components, leaving only the base band component.

Remark: Note that we have implicitely used rectangular pulse shaping since a_{11} and a_{12} are constant during the symbol interval. Since the integration effectively works as a matched filter (here pulse shaping is  $\Rightarrow$ a sinc in frequency domain)

Now, introduce a general pulse shaping filter $h(t)$. The modulator can then be depicted as



– FIG 2 –

Select, $\varphi_1(t) = h(t) \cos w_0 t$, $\varphi_2(t) = h(t) \sin w_0 t$ as basis functions. Then the orthogonality condition

$$\int_0^T \varphi_i \varphi_j \, dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

implies that

$$\int_0^T h(t)^2 \cos^2 w_0 t \; dt = 1$$

$$\int_0^T h(t)^2 \cos w_0 t \, \sin w_0 t \; dt = 0$$

which will be true, should $w_0 T \gg 1$.

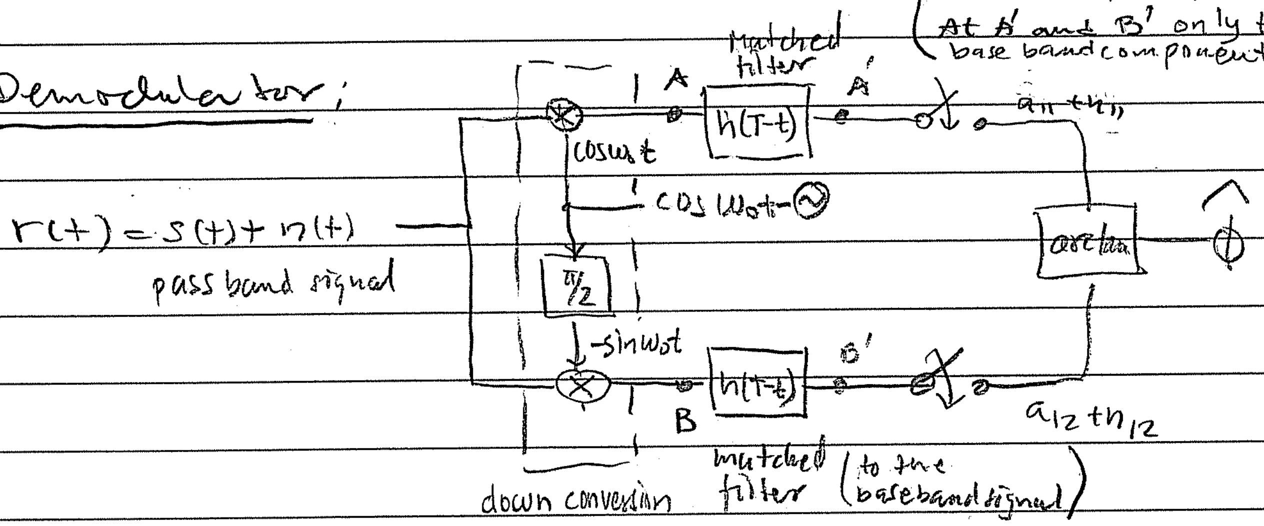
If the bandwidth, say B, of $h(t)$ is such that $B \ll w_0$ then $h(t)^2 \simeq$ const over a carrier period, over which $\cos w_0 t$ and $\sin w_0 t$ are orthogonal. The simplest pulse shape satisfying this is $\sqrt{\frac{2}{T}}$ $0 < t < T$, as we used above, but there are, of course, others, e.g. square root raise cosine.

The integrals in (*) are not necessarily efficient from an implementation perspective. It may be more attractive to use the following representation

(at point A and B we have baseband and double frequency components. At A' and B' only the base band component remains)

Modulator: As in Fig 2.

Demodulator:



Pulse shaping: $h(t) = \sqrt{\frac{2}{T}}$ rectangular pulse shaping

$$h * h = 2W_0 (\operatorname{sinc} W_0 t) \frac{\cos 2\pi (W - W_0) t}{1 - [4(W - W_0 t]^2}$$ ~~raised cosine~~ (RC) *)

($h(t) = \text{sqrt } RC$)

Matched filters can be efficiently implemented in discrete form. see p. 185 in Sklar.

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*) A square root cosine pulse shaping filter $h(t)$ is readily obtained by taking the square root of the frequency domain representation of the raised cosine and the inverse transform to the time domain

Complex envelope description

Modulation & demodulation can be describet by the use of complex notation

Any real band pass form $s(t)$ can be expressed as

$$s(t) = \mathrm{Re}\left\{ g(t)\, e^{j\omega_0 t} \right\}$$ – transmitted waveform

where $g(t)$ is the complex envelope (baseband message) and $e^{j\omega_0 t}$ is the carrier.

$$g(t) = x(t) + j\, y(t) = |g(t)|\, e^{j\theta(t)}$$

$$|g(t)| = \sqrt{x(t)^2 + y(t)^2} \qquad \theta(t) = \tan^{-1} \frac{y(t)}{x(t)}$$

Thus $s(t)$ can be expressed as

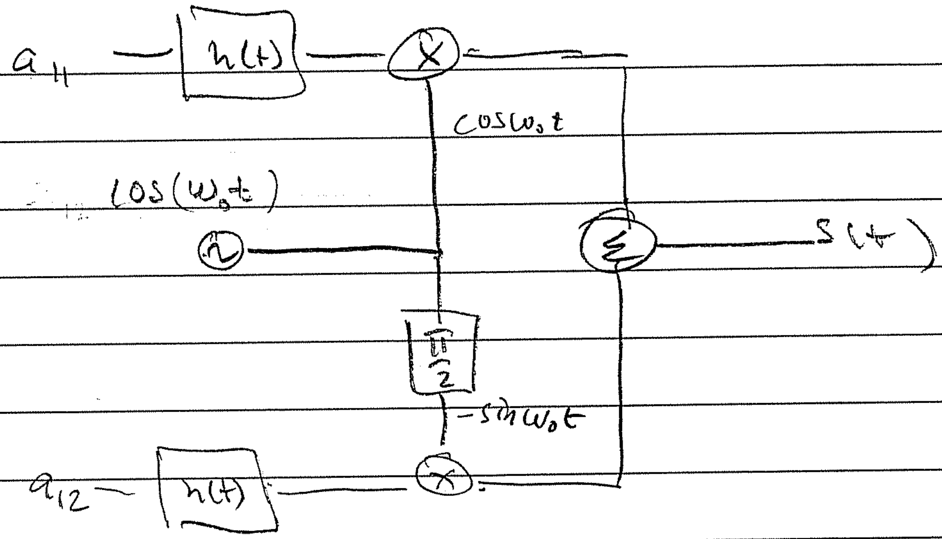
$$s(t) = \mathrm{Re}\left\{ (x(t) + j\, y(t))(\cos\omega_0 t + j \sin\omega_0 t) \right\}$$

$$= x(t)\cos\omega_0 t + j^2\, y(t) \sin\omega_0 t$$

$$= x(t)\cos\omega_0 t - y(t)\sin\omega_0 t$$

Let $x(t)$ and $y(t)$ be the pulse shaped sequences i.e. $x(t) = a_{11} h(t)$ $y(t) = a_{12} h(t)$ then by letting $\psi_1(t) = h(t)\cos(\omega_0 t)$ and $\psi_2(t) = -h(t)\sin(\omega_0 t)$ we can use the modulator demodulator structure

Modulator



Demodulator