EE-387 Probability for Electrical and Computer Engineers
Solution to Assignment 5

Problem 1: (Problem 4.8.7 from Yates and Goodman) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1; 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \le 1/4\}$. (a) What is the conditional PDF $f_{X,Y|A}(x,y)$? (b) What is $f_{Y|A}(y)$? (c) What is $E[Y|A]$? (d) What is $f_{X|A}(x)$? (e) What is $E[X|A]$?

Solutiont: (a) $A = \{Y \le 1/4\}$

$$P[A] = P[Y \le 1/4] = \int_{-1/2}^{1/2} \left(\int_0^{x^2} \frac{5x^2}{2} dy \right) dx + 2\int_{1/2}^{1} \left(\int_0^{1/4} \frac{5x^2}{2} dy \right) dx = \frac{19}{48}.$$

Therefore,

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{120x^2}{19} & (x,y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned} f_{Y|A}(y) &= \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)dx \\ &= \int_{-1}^{-\sqrt{y}} f_{X,Y|A}(x,y)dx + \int_{\sqrt{y}}^{1} f_{X,Y|A}(x,y)dx \\ &= 2\int_{\sqrt{y}}^{1} f_{X,Y|A}(x,y)dx \\ &= \frac{80}{19}(1-y^{3/2}). \end{aligned}$$

Therefore,

$$f_{Y|A}(y) = \begin{cases} \frac{80}{19}(1-y^{3/2}) & 0 \le y \le 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

(c) The conditional expectation of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)dy = \int_0^{1/4} y \frac{80}{19}(1-y^{3/2})dy = \frac{65}{532}.$$

(d)

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)dy.$$

Since limits will depend on the values of x, we consider for $-1/2 \leq x \leq 1/2$,

$$f_{X|A}(x) = \int_0^{x^2} \frac{120}{19} x^2 dy = \frac{120}{19} x^4$$

and for $-1 \leq x < -1/2$ or $1/2 < x \leq 1$,

$$f_{X|A}(x) = \int_0^{1/4} \frac{120}{19} x^2 dy = \frac{30}{19} x^2.$$

Therefore,

$$f_{X|A}(x) = \begin{cases} \frac{30}{19}x^2 & -1 \leq x < -1/2, \\ \frac{120}{19}x^4 & -1/2 \leq x \leq 1/2, \\ \frac{30}{19}x^2 & 1/2 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(e)

$$E[X|A] = \int_{-1}^{-1/2} x \frac{30}{19} x^2 dx + \int_{-1/2}^{1/2} x \frac{120}{19} x^4 dx + \int_{1/2}^{1} x \frac{30}{19} x^2 dx = 0.$$

Problem 2: (Problem 4.9.4 from Yates and Goodman) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the PDF $f_Y(y)$, the conditional PDF $f_{X|Y}(x|y)$, and the conditional expected value $E[X|Y=y]$.

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_y^1 2 dx = 2(1-y).$$

Therefore,

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is given $Y = y$, X is uniformly distributed in $[y, 1]$. The conditional expected value is

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx = \int_y^1 x \frac{1}{1-y} dx = \frac{1+y}{2}.$$

Problem 3: (Problem 4.11.2 from Yates and Goodman) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}.$$

(a) What are $E[X]$ and $E[Y]$? (b) Find ρ, the correlation coefficient of X and Y. (c) What are Var$[X]$ and Var$[Y]$? (d) What is the constant c? (e) Are X and Y independent?

Solution: (a) By matching the given joint PDF with the joint PDF of bivariate Gaussian, we obtain the following equations

$$\left(\frac{x-E[X]}{\sigma_X}\right)^2 = 4(1-\rho^2)x^2$$

$$\left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = 8(1-\rho^2)y^2$$

$$\frac{2\rho}{\sigma_X\sigma_Y} = 8(1-\rho^2).$$

The first two equations give $E[X] = E[Y] = 0$.

(b) The find the correlation coefficient ρ, we observe that

$$\sigma_X = 1/\sqrt{4(1-\rho^2)} \quad \sigma_Y = 1/\sqrt{8(1-\rho^2)}.$$

Using σ_X and σ_Y in the third equation yields $\rho = 1/\sqrt{2}$.

(c) Since $\rho = 1/\sqrt{2}$, now we can solve for σ_X and σ_Y and get

$$\sigma_X = 1/\sqrt{2} \qquad \sigma_Y = 1/2.$$

(d) From here we can solve for c as

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} = \frac{2}{\pi}$$

(e) X and Y are dependent because $\rho \neq 0$.

Problem 4: (Problem 4.11.8 from Yates and Goodman) Let X_1 and X_2 have a bivariate Gaussian PDF with correlation coefficient ρ_{12} such that each X_i is a Gaussian random variable with mean μ_i and variance σ_i^2. Show that $Y = X_1X_2$ has variance

$$\text{Var}[Y] = \sigma_1^2\sigma_2^2(1+\rho_{12}^2) + \sigma_1^2\mu_2^2 + \mu_1^2\sigma_2^2 - \mu_1^2\mu_2^2.$$

Hints: Use the iterated expectation to calculate

$$E[X_1^2X_2^2] = E[E[X_1^2X_2^2|X_2]].$$

Solution: Omitted.

Problem 5: (Problem 6.2.3 from Yates and Goodman) Random variables X and Y are independent exponential with expected values $E[X] = 1/\lambda$ and $E[Y] = 1/\mu$. If $\mu \neq \lambda$, what is the PDF of $W = X+Y$? If $\mu = \lambda$, what is $f_W(w)$?

Solution: PDF of W can be obtained by convolving two exponential distributions. (a) For $\lambda \neq \mu$,

$$\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(t) f_Y(w-t)dt \\
&= \int_0^w \lambda e^{-\lambda t}\mu e^{-\mu(w-t)}dt \\
&= \lambda\mu e^{-\mu w}\int_0^w e^{-(\lambda-\mu)t}dt = \begin{cases} \frac{\lambda\mu}{\lambda-\mu}[e^{-\mu w} - e^{-\lambda w}] & w \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) for $\lambda = \mu$, previous expression becomes invalid because zero will appear in the denominator. However, PDF of W can again be obtained similarly as

$$\begin{aligned}
f_W(w) &= \int_0^w \lambda e^{-\lambda t}\lambda e^{-\lambda(w-t)}dt \\
&= \lambda^2 e^{-\lambda w}\int_0^w dt \\
&= \begin{cases} \lambda^2 w e^{-\lambda w} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that when $\mu = \lambda$, W is the sum of two independent, identically distributed exponential RVs and has a second order Erlang PDF.

Problem 6: (Problem 6.2.5 from Yates and Goodman) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PDF of $W = X + Y$?

Solution: Recall

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)dx$$

where the integration is along the line $y = w - x$. Note since $W = X + Y$, we have the range for W as $S_W = \{w|0 \leq w \leq 2\}$. For $0 \leq w \leq 1$,

$$f_W(w) = \int_{w/2}^{w} 8x(w-x)dx = \frac{2w^3}{3}.$$

For $1 \leq w \leq 2$,

$$f_W(w) = \int_{w/2}^{1} 8x(w-x)dx = 4w - \frac{8}{3} - \frac{2w^3}{3}.$$

Therefore, the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} \frac{2w^3}{3} & 0 \leq w \leq 1, \\ 4w - \frac{8}{3} - \frac{2w^3}{3} & 1 \leq w \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 7: (a) If X is an Erlang (n, λ) random variable, show that the moment generating function of X is given by

$$\phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n.$$

(b) If X is a Gaussian random variable with mean μ and variance σ^2, show that the moment generating function of X is given by

$$\phi_X(s) = e^{s\mu + s^2\sigma^2/2}.$$

Solution: (a) The PDF for an Erlang (n, λ) is

$$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!},\ x \geq 0.$$

The MGF of X is just

$$\begin{aligned}
\phi_X(s) &= E[e^{sX}] \\
&= \int_0^\infty e^{sx} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} dx \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty x^{n-1} e^{-(\lambda-s)x} dx \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{(\lambda-s)^n} \underbrace{\int_0^\infty t^{n-1} e^{-t} dt}_{\Gamma(n)=(n-1)!} \\
&= \left(\frac{\lambda}{\lambda-s}\right)^n.
\end{aligned}$$

(b) The PDF for an Gaussian with mean μ and variance σ^2 is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

The MGF of X is just

$$\begin{aligned}
\phi_X(s) &= E[e^{sX}] \\
&= \int_{-\infty}^\infty e^{sx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left[-\frac{x^2 - 2(\mu+\sigma^2 s)x + \mu^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left[-\frac{x^2 - 2(\mu+\sigma^2 s)x + (\mu+\sigma^2 s)^2 - (\mu+\sigma^2 s)^2 + \mu^2}{2\sigma^2}\right] dx \\
&= \exp[\mu s + \sigma^2 s^2/2] \underbrace{\int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-(\mu+\sigma^2 s))^2}{2\sigma^2}\right] dx}_{=1} \\
&= e^{s\mu + s^2\sigma^2/2}.
\end{aligned}$$

Problem 8: Let X be a Gaussian random variable with mean zero and variance σ^2. Use the moment generating function to show that

$$E[X] = 0, \;\; E[X^2] = \sigma^2, \;\; E[X^3] = 0, \;\; E[X^4] = 3\sigma^4.$$

What can you say about $E[X^n]$ for arbitrary integer values of n?

<u>Solution</u>: From Problem 7, the moment generating function for X is

$$\phi_X(s) = e^{s^2\sigma^2/2}.$$

Therefore,

$$E[X] = \left.\frac{d\phi_X(s)}{ds}\right|_{s=0} = \left.e^{s^2\sigma^2/2}(s\sigma^2)\right|_{s=0} = 0.$$

$$E[X^2] = \left.\frac{d^2\phi_X(s)}{ds^2}\right|_{s=0} = \left.\sigma^2 e^{s^2\sigma^2/2} + (s^2\sigma^4)e^{s^2\sigma^2/2}\right|_{s=0} = \sigma^2.$$

Continuing in this manner and we can show that

$$E[X^3] = \left.(3\sigma^4 s + \sigma^6 s^3)e^{\sigma^2 s^2/2}\right|_{s=0} = 0$$

and

$$E[X^4] = \left.(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4)e^{\sigma^2 s^2/2}\right|_{s=0} = 3\sigma^4.$$

In general, one can deduce

$$E[X^n] = \begin{cases} 0 & n = 2k+1, \\ (1)(3)\cdots(n-1)\sigma^n & n = 2k. \end{cases}$$