X is a continuous RV defined on the basis of a fair coin flip as follows:

- If flip is heads, X is drawn at random from the interval $[-21, -9]$ according to the Uniform[-21,-9] distribution.
- If flip is tails, X is drawn at random from the interval $[0, \infty)$ according to the exponential density $a \exp(-ax)u(x)$ with parameter $a = 1/20$.

Problem 1: What are the values of $E[X|X < 0]$ and $E[X|X \geq 0]$?

Solution.

$$E[X|X < 0] = (-21 - 9)/2 = -15.$$

$$E[X|X \geq 0] = 1/a = 20.$$

Problem 2: What are the values of $\mathrm{Var}[X|X < 0]$ and $\mathrm{Var}[X|X \geq 0]$?

Solution.

$$\mathrm{Var}[X|X < 0] = (-9 - (-21))^2/12 = 12.$$

$$\mathrm{Var}[X|X \geq 0] = 1/a^2 = 400.$$

Problem 3: What are the values of $E[X^2|X < 0]$ and $E[X^2|X \geq 0]$?

Solution.

$$E[X^2|X < 0] = 12 + (-15)^2 = 237.$$

$$E[X^2|X \geq 0] = 400 + (20)^2 = 800.$$

Problem 4: What are the values of $E[X]$, $\mathrm{Var}[X]$?

Solution.

$$E[X] = 0.5 * (-15) + 0.5 * (20) = 2.5$$

$$E[X^2] = 0.5 * (237) + 0.5 * 800 = 518.5.$$

$$\mathrm{Var}[X] = 518.5 - (2.5)^2 = 512.25.$$

X is a continuous RV with PDF

$$f_X(x) = \frac{x}{18}, \quad 0 \le x \le 6 \text{ (zero elsewhere)}$$

Problem 5: $E[X] = ?$

Solution.

$$E[X] = \int_0^6 x(x/18)dx = 4.$$

Problem 6: $P(0 \le X \le 3) = ?$

Solution.

$$P(0 \le X \le 3) = \int_0^3 (x/18)dx = 1/4.$$

Problem 7: $E[X|0 \le X \le 3] = ?$

Solution.

$$E[X|0 \le X \le 3] = \frac{\int_0^3 x(x/18)dx}{1/4} = 2.$$

X is a discrete RV whose values are all integers. The CDF of X satisfies:

$$F_X(0.5) = 0, \quad F_X(1.5) = 0.20, \quad F_X(2.5) = 0.40$$

$$F_X(3.5) = 0.70, \quad F_X(4.5) = 0.85, \quad F_X(5.5) = 1$$

Problem 8: $P(2 \le X \le 4) = ?$

Solution.

$$P(2 \le X \le 4) = F_X(4) - F_X(1) = F_X(4.5) - F_X(1.5) = 0.65.$$

Problem 9: $E[X] = ?$

Solution.

$$\begin{aligned} p_X(1) &= 0.20 \\ p_X(2) &= 0.20 \\ p_X(3) &= 0.30 \\ p_X(4) &= 0.15 \\ p_X(5) &= 0.15 \end{aligned}$$

$$E[X] = [1, 2, 3, 4, 5] \bullet [.20, .20, .30, .15, .15] = 2.85.$$

Problem 10: $\mathrm{Var}[X] = ?$

Solution.

$$E[X^2] = [1, 4, 9, 16, 25] \bullet [.20, .20, .30, .15, .15] = 9.85.$$

$$\mathrm{Var}[X] = E[X^2] - \mu_X^2 = 9.85 - (2.85)^2 = 1.7275.$$

Problem 11: Given $\mu = 50$ and $\sigma = 10$, then

$$\int_{32.5}^{42.5} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = ?$$

Solution.

$$\Phi\left(\frac{42.5-50}{10}\right) - \Phi\left(\frac{32.5-50}{10}\right) = \Phi(1.75) - \Phi(0.75) = 0.187.$$

Problem 12: Find the real number C such that

$$\int_{C}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = 0.05.$$

Solution. We must have $\Phi(C) = 0.95$, so $C = 1.645$.

A, B are events satisfying:

$$P(A) = 2/3, \quad P(B|A) = 1/2, \quad P(B^c|A^c) = 1/3$$

Problem 13: $P(B) = ?$

Solution. The "Bayes array" is

$$\begin{array}{ccc} & B & B^c \\ A & 1/2 & 1/2 \\ A^c & 2/3 & 1/3 \end{array}$$

Multiply the first row by 2/3 and the second row by 1/3:

$$\begin{array}{ccc} & B & B^c \\ A & 1/3 & 1/3 \\ A^c & 2/9 & 1/9 \end{array}$$

Then, $P(B)$ is sum of the left column, which is 5/9.

Problem 14: $P(A^c|B) = ?$

Solution. Divide each column of the array by the column sum:

$$\begin{array}{ccc} & B & B^c \\ A & 3/5 & 3/4 \\ A^c & 2/5 & 1/4 \end{array}$$

Then, $P(A^c|B)$ is the element in the lower left corner, namely, 2/5.

Problem 15: $P(A|B^c) = ?$

Solution. $P(A|B^c)$ is the element in the upper right corner, namely, 3/4.

A_1, A_2, A_3, A_4, A_5 are independent events each having probability 0.90. Let E, F be the events

$$\begin{aligned} E &= A_1 \cap A_2^c \cap A_3 \\ F &= A_3 \cap A_4^c \cap A_5 \end{aligned}$$

Problem 16: $P(E) = P(F) = ?$

Solution. $(0.90)(0.10)(0.90) = 0.0810$.

Problem 17: $P(E \cap F) = ?$

Solution. $E \cap F$ is the event

$$A_1 \cap A_2^c \cap A_3 \cap A_4^c \cap A_5,$$

which has prob

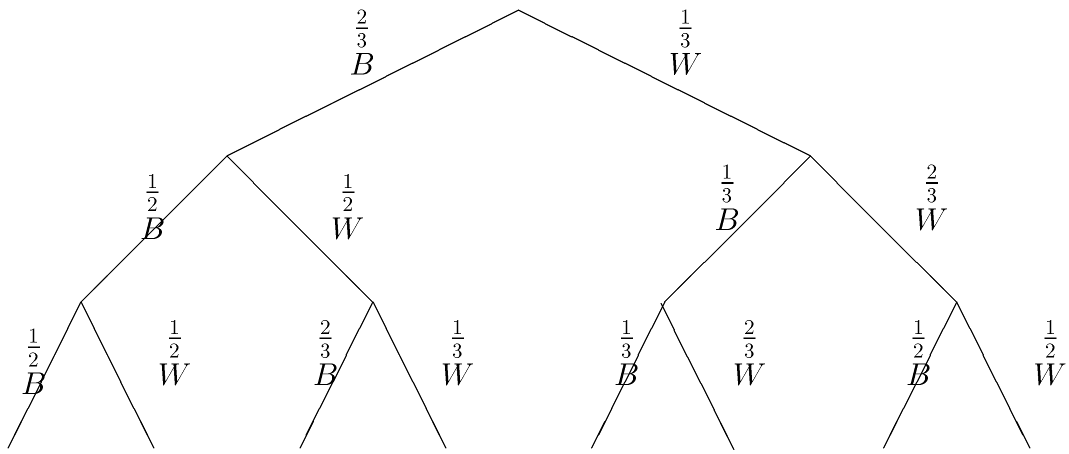
$$(0.90)(0.10)(0.90)(0.10)(0.90) = 0.0073.$$

Problem 18: $P(E \cup F) = ?$

Solution.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.1547.$$

An experiment consists of three random draws. On each draw, a black ball or a white ball is drawn. The tree of the experiment is below.



Problem 19: $P(\text{first draw is black}) = ?$

Solution. 2/3.

Problem 20: $P(\text{second draw is black}) = ?$

Solution.

$$P(BB) + P(WB) = 4/9.$$

Problem 21: $P(\text{third draw is black}) = ?$

Solution.

$$P(BBB) + P(BWB) + P(WBB) + P(WWB) = 29/54.$$