EE 3025 S2005 Exam 2 Solutions

Note: In each solution, I put the answer in simple enough form to be completely acceptable

PROBLEM 1

RV's X, Y satisfy the following properties:

$$E(Y|X) = 2X - 5, \quad \mu_X = 1, \quad \sigma_X = 3, \quad \sigma_Y = 18.$$

(a) Compute μ_Y using law of iterated expectation.

Solution.

$$E[Y] = E[E[Y|X]] = E[2X - 5] = 2E[X] - 5 = -3$$

(b) Compute $E[XY]$ using law of iterated expectation.

Solution.

$$E[XY] = E[XE[Y|X]] = E[2X^2 - 5X] = 2E[X^2] - 5E[X] = 2 * (1^2 + 3^2) - 5 = 15.$$

(c) Compute $\rho_{X,Y}$.

Solution.

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - \mu_X \mu_Y}{54} = \frac{18}{54} = 1/3.$$

PROBLEM 2

$\bar{X}_n$ denotes the sample mean of a sample of size n from a prob dist with unknown mean μ and stan dev $\sigma = 3$. In parts(a),(b) below, you are to find the smallest n you can such that

$$P[\mu - 0.05 \le \bar{X}_n \le \mu + 0.05] \ge 0.60.$$

(a) You are sampling from a Gaussian distribution. Find n. (Hint: $\Phi(0.84) - \Phi(-0.84) = 0.60$.)

Solution. We must have

$$\frac{k\sigma}{\sqrt{n}} = 0.05,$$

where $k = 0.84$. Solving for n, you get

$$n = [60 * 0.84]^2.$$

If you then use a calculator, you get $n = 2541$. The actual figure may vary slightly from this when you replace 0.84 by enough decimal places of accuracy to get the precise value of n.

(b) You are sampling from a nonGaussian distribution. Find n.

Solution. We must have

$$\frac{k\sigma}{\sqrt{n}} = 0.05,$$

where

$$1 - \frac{1}{k^2} = 0.60.$$

You get $k = \sqrt{2.5}$, and then

$$n = 9000.$$

PROBLEM 3

RV's T_1, T_2 are independent, each having the PDF

$$(0.6)\delta(t+2) + (0.4)\delta(t-1)$$

Let $T = T_1 - 2T_2$.

(a) Write down the PDF of $-2T_2$.

Solution. The original values are -2 and 1. Scaling by -2, the new values are 4 and -2. Therefore the PDF will become

$$(0.6)\delta(t-4) + (0.4)\delta(t+2)$$

(b) Find the PDF of T by convoluting the PDF of T_1 with the PDF of $-2T_2$. (Hint: $\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$.)

Solution. There are four "cross-product" convolution terms:

$$\begin{aligned}
(0.6)\delta(t+2) * (0.6)\delta(t-4) &= (0.36)\delta(t-2) \\
(0.6)\delta(t+2) * (0.4)\delta(t+2) &= (0.24)\delta(t+4) \\
(0.4)\delta(t-1) * (0.6)\delta(t-4) &= (0.24)\delta(t-5) \\
(0.4)\delta(t-1) * (0.4)\delta(t+2) &= (0.16)\delta(t+1)
\end{aligned}$$

Now add these up:

$$(0.24)\delta(t+4) + (0.16)\delta(t+1) + (0.36)\delta(t-2) + (0.24)\delta(t-5).$$

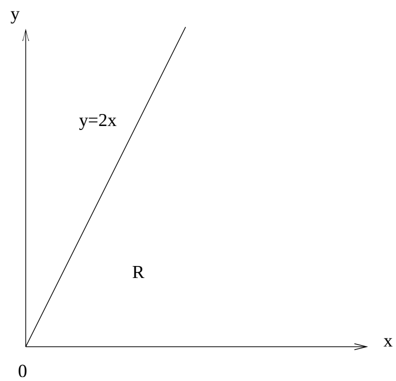
(c) Compute $P[-3 \le T \le 3]$.

Solution. You pick up the two middle spikes:

$$Answer = 0.16 + 0.36 = 0.52.$$

PROBLEM 4
R is the infinite region sketched below. RV's X, Y have the joint PDF

$$f_{X,Y}(x,y) = (0.5)\exp(-x), \quad (x,y) \in R \; (zero\ elsewhere)$$



(a) Tell me what the limits a_1, a_2 below are. (Do not do anything else.)

$$f_X(x) = \int_{a_1}^{a_2} (0.5)\exp(-x)dy$$

Solution.

$$a_1 = 0, \quad a_2 = 2x.$$

(b)Tell me what the limits a_3, a_4, a_5, a_6 are. (Do not do anything else.)

$$P[1 \le X \le 7 | Y = 3] = \frac{\int_{a_3}^{a_4} (0.5)\exp(-x)dx}{\int_{a_5}^{a_6} (0.5)\exp(-x)dx}$$

Solution.

$$a_3 = 1.5, \quad a_4 = 7, \quad a_5 = 1.5, \quad a_6 = \infty.$$

(c) Tell me what the limits b_1, b_2, b_3, b_4 are. (Do not do anything else.)

$$P[Y \le X] = \int_{b_1}^{b_2} \int_{b_3}^{b_4} (0.5)\exp(-x)dydx$$

Solution.

$$b_1 = 0, \quad b_2 = \infty, \quad b_3 = 0, \quad b_4 = x.$$

PROBLEM 5
Discrete RV's X, Y have the following joint PMF:

	$Y=3$	$Y=2$	$Y=1$
$X=1$	1/6	1/4	0
$X=2$	0	1/6	1/4
$X=3$	0	0	1/6

Given: $\mu_X = \mu_Y = 7/4$ and $\sigma_X^2 = \sigma_Y^2 = 25/48$.

(a) Compute $E[XY]$.
Solution.

$$E[XY] = (1*3)(1/6) + (1*2)(1/4) + (2*2)(1/6) + (2*1)(1/4) + (3*1)(1/6) = 8/3.$$

(b) Compute $Cov(X, Y)$.
Solution.

$$Cov(X, Y) = E[XY] - \mu_X \mu_Y = (8/3) - (7/4)^2 = (8/3) - (49/16)$$

(With calculator, this simplifies to -0.3958.)
(c) Compute $\rho_{X,Y}$.
Solution.

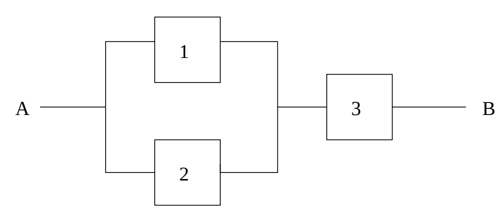
$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{(8/3) - (49/16)}{25/48} = (128/25) - (147/25) = -19/25 = -0.76.$$

PROBLEM 6
A relay circuit provides a connection from point A to point B and contains three switches 1,2,3. For $i = 1, 2, 3$, switch i has a random lifetime T_i which is exponentially distributed with mean lifetime $150i$ (hours).

(a) The overall random lifetime T_{AB} of the A to B connection is given by the formula

$$T_{AB} = \min(\max(T_1, T_2), T_3).$$

Draw a block diagram of the relay circuit.
Solution.



(b) For $i = 1, 2, 3$, let $p_i = P[T_i \geq t]$ for $t \geq 0$. Determine p_1, p_2, p_3 as functions of t. (Hint: See Formula Sheet for PDF of exponential distribution.)
Solution.

$$\begin{aligned} p_1 &= \exp(-t/150) \\ p_2 &= \exp(-t/300) \\ p_3 &= \exp(-t/450) \end{aligned}$$

(c) Let $R(t) = P[T_{AB} \geq t]$ for $t \geq 0$. Write down an expression for $R(t)$ as a function of p_1, p_2, p_3.

Solution.

$$R(t) = [1 - (1 - p_1)(1 - p_2)]p_3.$$

PROBLEM 7

RV's X, Y have joint PDF of the form

$$f_{X,Y}(x, y) = C \exp[-0.5(x^2 - xy + 2y^2)],$$

where C is a positive constant.

(a) Compute $E[Y|X = x]$ and $E[X|Y = y]$.

Solution. To find $E[Y|X = x]$, you can complete the square on y in the exponent, which is the same thing as setting $\partial/\partial y$ of the exponent equal to zero and solving for y:

$$\frac{\partial}{\partial y}(x^2 - xy + 2y^2) = -x + 4y = 0.$$

$$y = x/4,$$

and so

$$E[Y|X = x] = x/4.$$

To find $E[X|Y = y]$, you can complete the square on x in the exponent, which is the same thing as setting $\partial/\partial x$ of the exponent equal to zero and solving for x:

$$\frac{\partial}{\partial x}(x^2 - xy + 2y^2) = 2x - y = 0.$$

$$x = y/2,$$

and so

$$E[X|Y = y] = y/2.$$

(b) Compute $Var(Y|X = x)$ and $Var(X|Y = y)$.

Solution. When you complete the square on y, the $(y - condmean)^2$ term will have denominator $1/2$. Therefore,

$$Var(Y|X = x) = 1/2.$$

When you complete the square on x, the $(x - condmean)^2$ term will have denominator 1. Therefore,

$$Var(X|Y = y) = 1.$$

(c) Compute $\rho_{X,Y}$, σ_X^2, σ_Y^2.

Solution. The coefficient of x in $E[Y|X = x]$ will always be $\rho\sigma_Y/\sigma_X$ and the coefficient of y in $E[X|Y = y]$ will always be $\rho\sigma_X/\sigma_Y$. This immediately gives us the equations

$$\begin{aligned} \frac{\rho\sigma_Y}{\sigma_X} &= 1/4 \\ \frac{\rho\sigma_X}{\sigma_Y} &= 1/2 \end{aligned}$$

Multiplying the two equations together, we see that

$$\rho^2 = 1/8.$$

The slope of each of the straight line "regression of the mean functions" $E[X|Y = y]$ and $E[Y|X = x]$ is positive, so ρ must be positive. We conclude that

$$\rho = \frac{1}{\sqrt{8}}.$$

We have

$$Var(Y|X = x) = 1/2 = \sigma_Y^2(1 - \rho^2) = \sigma_Y^2(7/8).$$

Therefore,

$$\sigma_Y^2 = 4/7.$$

We also have

$$Var(X|Y = y) = 1 = \sigma_X^2(1 - \rho^2) = \sigma_X^2(7/8).$$

Therefore,

$$\sigma_X^2 = 8/7.$$

You can also solve part(c) by inverting a 2×2 matrix (which is less exciting, but still OK).

PROBLEM 8

RV's X, Y have the SAME variance σ^2.
Given: $Var(X + Y) = 100$ and $Var(X - Y) = 1700$.

(a) Compute σ^2.
Solution.

$$\begin{aligned} Var(X + Y) &= Var(X) + Var(Y) + 2Cov(X, Y) = 2\sigma^2 + 2Cov(X, Y) = 100 \\ Var(X - Y) &= Var(X) + Var(Y) - 2Cov(X, Y) = 2\sigma^2 - 2Cov(X, Y) = 1700 \end{aligned}$$

Adding the two equations together and dividing by 4, you conclude that

$$\sigma^2 = 450.$$

(b) Compute $Cov(X, Y)$ and $\rho_{X,Y}$.

Solution. Plugging $\sigma^2 = 450$ back into one of the equations from part(a) solution will tell you that

$$Cov(X, Y) = -400.$$

Dividing this by σ^2 will give the correlation coefficient. We conclude that

$$\rho = -\frac{8}{9}.$$

(c) Compute $Cov(X + 2Y, X - 2Y)$.
Solution.

$$Cov(X + 2Y, X - 2Y) = Cov(X, X) - 4Cov(Y, Y) = -3\sigma^2 = -1350.$$