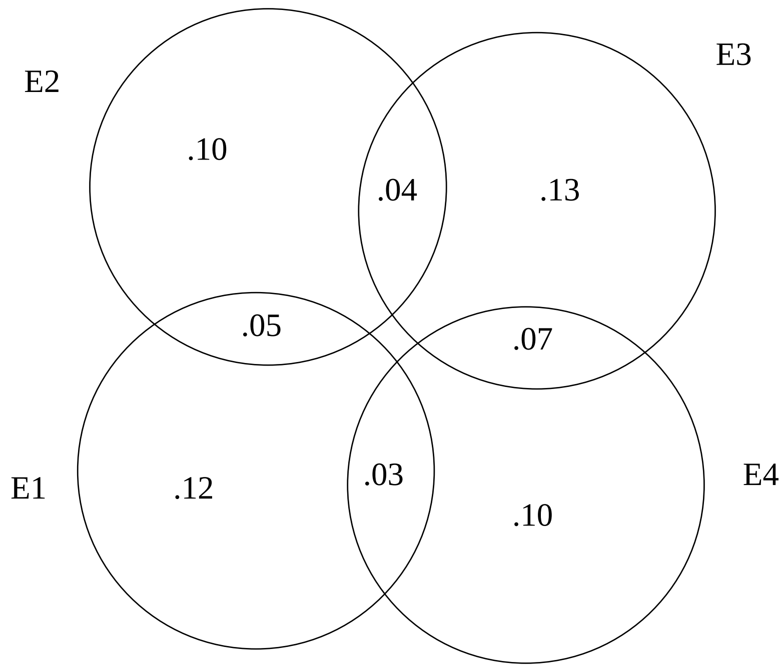
EE 3025 S2005 Final Exam Solutions

PROBLEM 1: In Venn Diagram below, events E_1 through E_4 are each everything inside the indicated circle. Each E_i is the union of 3 disjoint pieces with the given probabilities. Answer the questions below.



(a) Find P[none of E_i's occur].

Solution. Add up the eight probs in the diagram and subtract from 1. Answer is 0.36.

(b) Find P[exactly one of E_i's occurs].

Solution. By inspection, it's

$$.13 + .10 + .12 + .10 = .45.$$

(c) Find P[exactly one E_i occurs | at least one E_i occurs].

Solution. Take answer to (b) and divide by $1-$ answer to (a). The answer is $45/64$.

(d) Find P[at least one E_i occurs | E_4 does not occur].

Solution. First, compute prob at least one E_i occurs and E_4 does not occur. By inspection of the diagram, this is

$$.12 + .13 + .05 + .10 + .04 = .44$$

Divide this by prob E_4 does not occur, which is .8. The answer is $44/80$.

PROBLEM 2: For events E, A, B, we are given $P(E) = 0.7$ and the cond prob's in the table below.

$$\begin{array}{c} \\ E \\ E^c \end{array} \begin{pmatrix} A \cap B & A \cap B^c & A^c \cap B & A^c \cap B^c \\ P(A \cap B|E) = 0.3 & P(A \cap B^c|E) = 0.3 & P(A^c \cap B|E) = 0.1 & P(A^c \cap B^c|E) = \\ P(A \cap B|E^c) = 0.4 & P(A \cap B^c|E^c) = 0.2 & P(A^c \cap B|E^c) = 0.2 & P(A^c \cap B^c|E^c) = \end{pmatrix}$$

(a) Fill in the two missing cond probs in the above table.

Solution. The row sums must each be 1. So the two missing probs are .3, .2.

(b) Compute the four probs asked for in the following table:

$$\begin{array}{c} \\ A \\ A^c \end{array} \begin{pmatrix} B & B^c \\ P(A \cap B) = ? & P(A \cap B^c) = ? \\ P(A^c \cap B) = ? & P(A^c \cap B^c) = ? \end{pmatrix}$$

Solution. Take [.7 .3] times the matrix at the beginning of the problem. You get

$$P(A \cap B) = .33, \quad P(A \cap B^c) = .27, \quad P(A^c \cap B) = .13, \quad P(A^c \cap B^c) = .27.$$

(c) Compute $P(A), P(B)$.

Solution. To get $P(A)$ and $P(B)$, you simply sum the first row and first column of matrix obtained in (b). You get

$$P(A) = .6, \quad P(b) = .46.$$

(d) Compute $P(E|A \cap B)$, $P(A|B)$.

Solution. We have

$$P(E \cap A \cap B \cap E) = P(A \cap B|E)P(E) = .21$$

$$P(E^c \cap A \cap B) = P(A \cap B)P(E^c) = .12$$

To convert these to "backward cond probs" $P(E|A \cap B)$ and $P(E^c|A \cap B)$, normalize getting

$$P(E|A \cap B) = .21/.33, \quad P(E^c|A \cap B) = .12/.33.$$

To get $P(A|B)$ and $P(A^c|B)$, normalize 1st column of matrix in (b), getting

$$P(A|B) = .33/.46, \quad P(A^c|B) = .13/.46.$$

PROBLEM 3: This problem concerns the binomial distribution.

(a) The first three rows of Pascal's triangle are the following:

1 1
1 2 1
1 3 3 1

Propogate Row 4 through Row 8 of Pascal's triangle. (The first two entries of Row 8 should be $1, 8$.)

Solution.

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          1     4     6     4     1
       1     5    10    10     5     1
    1     6    15    20    15     6     1
  1    7    21    35    35    21     7     1
1    8    28    56    70    56    28     8     1
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(b) Let RV X be the number of heads obtained in 8 tosses of a fair coin. Compute the probability that $\log_2 X$ is an integer.

Solution. X is either 1,2,4, or 8. Using last row of triangle, the answer is therefore

$$(8 + 28 + 70 + 10)/2^8 = 107/256.$$

(c) Bill tosses a fair coin 4 times. Joe tosses a fair coin 5 times. Compute the probability that Joe obtains exactly two more heads than Bill.

Solution. You need positions 1,2,3,4 in row 4 of triangle plus positions 3,4,5,6 in row 5 of triangle:

$$[1, 4, 6, 4] \cdot [10, 10, 5, 1]/(2^4 * 2^5) = 84/512.$$

PROBLEM 4: Let $X(t)$ be the Poisson process with arrival rate

(a) Compute $E[X(2)]$ and $E[X(3)]$.

Solution. $E[X(t)] = \lambda t$, so the answers are 3 and 4.5.

(b) Compute $E[X(2)X(3)]$.

Solution.

$$Cov(X(2), X(3)) = Cov(X(2), X(2)) + Cov(X(2), X(3) - X(2)) = Var(X(2)) + 0 = 3.$$

Therefore,

$$E[X(2)X(3)] = Cov(X(2), X(3)) + E[X(2)]E[X(3)] = 3 + 3(4.5) = 16.5.$$

(c) Compute $P[X(3) - X(2) = 1|X(2) = 3]$.

Solution. By independent increments, this is the same as $P[X(3) - X(2) = 1]$, which is $\exp(-1.5) * 1.5$.

(d) Compute $P[X(2) - X(1) = 1|X(2) = 3]$.

Solution.

$$\begin{aligned} P[X(2) - X(1) = 1|X(2) = 3] &= \frac{P[X(2) - X(1) = 1,\ X(2) = 3]}{P[X(2) = 3]} \\ &= \frac{P[X(1) = 2,\ X(2) - X(1) = 1]}{P[X(2) = 3]} \\ &= \frac{P[X(1) = 2]P[X(2) - X(1) = 1]}{P[X(2) = 3]} \\ &= \frac{\exp(-\lambda)(\lambda^2/2)\exp(-\lambda)\lambda}{\exp(-2\lambda)(2\lambda)^3/6} = 3/8 \end{aligned}$$

PROBLEM 5: A continuous-time Gaussian ergodic WSS process $X(t)$ has mean $\mu_X < 0$ and autocorrelation function

$$R_X(\tau) = 96(0.25)^{|\tau|} + 16.$$

(a) Compute μ_X, P_X, σ_X^2.

Solution. Since we have an ergodic Gaussian process, μ_X^2 is the limit of $R_X(\tau)$ as $\tau \to \infty$. Therefore,

$$\mu_X = \sqrt{16} = -4.$$

Also,

$$P_X = R_X(0) = 112,$$

and

$$\sigma_X^2 = P_X - \mu_X^2 = 112 - 16 = 96.$$

(b) Compute $Cov[X(4), X(6)]$.

Solution. The answer is $R_X(2) - \mu_X^2$, which is 6.

(c) Compute the mean and variance of the Gaussian RV $Y = X(6) - (0.5)X(4)$.

Solution.

$$\mu_Y = E[X(6)] - 0.5E[X(4)] = -2.$$

$$\sigma_Y^2 = Var(X(6)) + 0.25Var(X(4)) - Cov(X(4), X(6)) = 114.$$

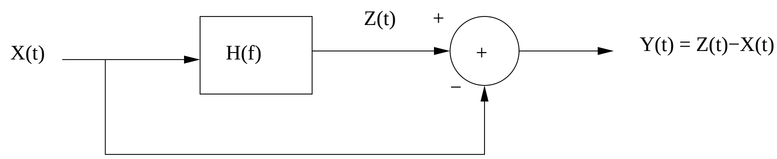
(d) Find the constant C such that

$$P[Y \geq 112] = \int_C^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) dz.$$

Solution. $Z = (Y - \mu_Y)/\sigma_Y = (Y + 2)/\sqrt{114}$ has the standard Gaussian distribution. Therefore

$$P[Y \geq 112] = P[Z \geq 114/\sqrt{114} = \sqrt{114}].$$

Therefore, $C = \sqrt{114}$.



PROBLEM 6: In the block diagram above, $X(t)$ is a continuous-time WSS random signal with power spectral density

$$S_X(f) = 100|f|, \quad -1 \leq f \leq 1 \ (zero\ elsewhere),$$

and the freq response function $H(f)$ is defined by

$$H(f) = (1.5)f^2, \quad -1 \leq f \leq 1 \ (zero\ elsewhere).$$

(a) Compute P_X.

Solution.

$$P_X = \int_{-1}^{1} 100|f|df = 2\int_0^1 100f df = 100.$$

(b) Determine $S_Z(f)$ and use it to compute P_Z.

Solution.

$$S_Z(f) = |H(f)|^2 S_X(f) = 225f^4|f|, \quad -1 \leq f \leq 1 \ (zero\ elsewhere).$$

$$P_Z = \int_{-1}^{1} 225f^4|f|df = 450\int_0^1 f^5 df = 75.$$

(c) Determine $S_Y(f)$ using the frequency response of the LTI system that takes $X(t)$ into $Y(t)$. Then use $S_Y(f)$ to compute P_Y.

Solution. The overall impulse response is $h(t) - \delta(t)$, and so the overall freq response function is $H(f) - 1$. Therefore,

$$P_Y = \int |H(f) - 1|^2 S_X(f)df = 200\int_0^1 (1.5f^2 - 1)^2 f df = 25.$$

PROBLEM 7: A discrete-time WSS process X_n has autocorrelation function

$$R_X(\tau) = 9\delta[\tau] + 6\delta[\tau+1] + 6\delta[\tau-1] + 5\delta[\tau+2] + 5\delta[\tau-2].$$

(a) Find the constant A such that $E[(X_3 - AX_2)^2]$ is a minimum.

Solution.

$$A = \frac{R_X(1)}{R_X(0)} = 2/3.$$

(b) Find the constants B, C such that $E[(X_3 - BX_2 - CX_1)^2]$ is a minimum.

Solution.

$$\begin{bmatrix} R_X(0) = 9 & R_X(1) = 6 \\ R_X(1) = 6 & R_X(0) = 9 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} R_X(1) = 6 \\ R_X(2) = 5 \end{bmatrix}.$$

The solutions are

$$B = 8/15, \quad C = 1/5.$$

PROBLEM 8: In this problem, you compute signal-to-noise ratios. The input to a filter is $X(t)+Z(t)$, where $X(t)$, the "signal", is CT WSS with power spectrum

$$S_X(f) = 2(100-|f|), \ \ -100 \le f \le 100 \ (\textit{zero elsewhere}),$$

and $Z(t)$, the "noise", is CT WSS with power spectrum

$$S_Z(f) = 1, \ \ -100 \le f \le 100 \ (\textit{zero elsewhere}).$$

The filter frequency response function is

$$H(f) = 100-|f|, \ \ -100 \le f \le 100 \ (\textit{zero elsewhere}).$$

(a) Compute the filter input signal-to-noise ratio P_X/P_Z.

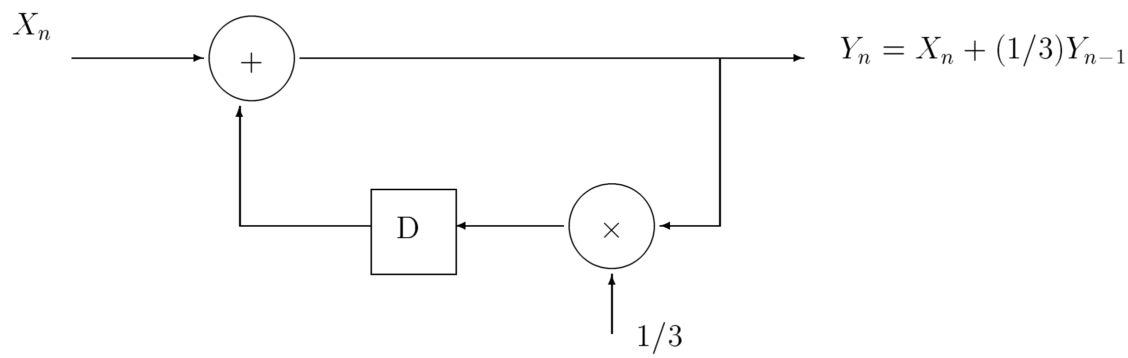
Solution.

$$\frac{P_X}{P_Z} = \frac{\int_{-100}^{100} 2(100-|f|)df}{\int_{-100}^{100} df} = 100.$$

(b) Compute the filter output signal-to-noise ratio P_{X^0}/P_{Z^0}, where $X^0(t)$ is the response of the filter to $X(t)$ and $Z^0(t)$ is the response of the filter to $Z(t)$.

Solution.

$$\frac{P_{X^0}}{P_{Z^0}} = \frac{\int |H(f)|^2 S_X(f)df}{\int |H(f)|^2 S_Z(f)df} = \frac{\int_{-100}^{100} 2(100-|f|)^3 df}{\int_{-100}^{100} (100-|f|)^2 df} = 150.$$



PROBLEM 9: This problem concerns the first order autoregressive DT linear system given above.

(a) Take the random input sequence to be IID with each X_n chosen equiprobably from the set

$$\{1, 2, 3, 4, 5\}.$$

Compute μ_X and μ_Y.

Solution.

$$\mu_X = (1 + 2 + 3 + 4 + 5)/5 = 3.$$

Taking expected value of both sides of $Y_n = X_n + (1/3)Y_{n-1}$, you get the equation

$$\mu_Y = \mu_X + (1/3)\mu_Y,$$

from which you easily deduce that $\mu_Y = 4.5$.

(b) Take the random input sequence to be IID with each X_n chosen equiprobably from the set

$$\{-2, -1, 0, 1, 2\}.$$

Compute P_X and P_Y.

Solution.

$$P_X = [(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2]/5 = 2.$$

Taking expected value of square of both sides of $Y_n = X_n + (1/3)Y_{n-1}$, you get

$$P_Y = P_X + (1/9)P_Y + 2E[X_n Y_{n-1}].$$

Because X_n is white noise, the last term on the right is zero. From the equation

$$P_Y = P_X + (1/9)P_Y,$$

you easily deduce that $P_Y = 9/4$.

PROBLEM 10: $X(t)$ is a continuous-time WSS process with mean $\mu_X = 2$ and autocorrelation function $R_X(\tau) = 4 + 3\delta(\tau)$. Let Y be the random variable

$$Y = \int_0^\pi (\sin t) X(t) dt.$$

(a) Compute μ_Y.

Solution.

$$\begin{aligned} \mu_Y &= E[\int_0^\pi (\sin t) X(t) dt] \\ &= \int_0^\pi E[(\sin t) X(t)] dt \\ &= \int_0^\pi (\sin t) E[X(t)] dt \\ &= \int_0^\pi 2 \sin t dt = 4 \end{aligned}$$

(b) Compute σ_Y^2. (Hint: The "double integral trick" works nicely here.)

Solution.

$$\begin{aligned} E[Y^2] &= \int_0^\pi \int_0^\pi (\sin s)(\sin t) R_X(s-t) ds dt \\ &= \int_0^\pi \int_0^\pi (\sin s)(\sin t)[4 + 3\delta(s-t)] ds dt \\ &= 4(\int_0^\pi \sin t dt)^2 + 3 \int_0^\pi \sin^2 t dt \\ &= 16 + 3\pi/2 \end{aligned}$$

Subtracting off $\mu_Y^2 = 16$, we see that $\sigma_Y^2 = 3\pi/2$.

PROBLEM 11: X_n, the "signal process", is DT WSS with 0 mean and autocorrelation function

$$R_X(\tau) = 3(2^{-|\tau|}).$$

Z_n, the "noise process", is DT WSS with mean 2 and autocorrelation function

$$R_Z(\tau) = 2\delta[\tau] + 4.$$

Signal and noise are orthogonal:

$$E[X_m Z_n] = 0, \quad all\ m, n.$$

We let process $Y_n = X_n + Z_n$ be the sum of the signal and noise processes.

(a) Use orthogonality principle to find constant A which makes $E[(X_n - AY_n)^2]$ a minimum.

Solution. The OP says

$$E[(X_n - AY_n)Y_n] = 0.$$

This gives

$$A = \frac{E[X_n Y_n]}{E[Y_n^2]} = \frac{R_X(0)}{R_Y(0)} = \frac{R_X(0)}{R_X(0) + R_Z(0)} = 1/3.$$

(b) Use orthogonality principle to find constants B, C which make $E[(X_n - BY_n - C)^2]$ a minimum.

Solution. The OP says

$$\begin{aligned} E[(X_n - BY_n - C)Y_n] &= 0 \\ E[X_n - BY_n - C] &= 0 \end{aligned}$$

These reduce to

$$\begin{aligned} R_Y(0)B + \mu_Y C &= R_X(0) \\ \mu_Y B + C &= \mu_X \end{aligned}$$

and therefore

$$\begin{aligned} 9B + 2B &= 3 \\ 2B + C &= 0 \end{aligned}$$

Solving, you get $B = 3/5$ and $C = -6/5$.

PROBLEM 12: $X(t)$ is the continuous-time process

$$X(t) = 5 + A\cos(2\pi t + \Theta_1),$$

where A is a RV equiprobable over the values $1, 2$ and where Θ_1 is independent of A and uniformly distributed in $[0, 2\pi]$. Let $Y(t)$ be the continuous-time process

$$Y(t) = X(t)\cos(20\pi t + \Theta_2),$$

where Θ_2 is uniformly distributed in $[0, 2\pi]$ and independent of (A, Θ_1).

(a) $X(t)$ is WSS. Specify $R_X(\tau)$, plot $S_X(f)$, and then use these results to compute P_X two different ways.

Solution. Since $E[A^2] = 2.5$, the random sinusoid component of $X(t)$ has autocorrelation function

$$(E[A^2]/2)\cos(2\pi\tau) = (1.25)\cos(2\pi\tau).$$

Therefore,

$$R_X(\tau) = 25 + (1.25)\cos(2\pi\tau),$$

and

$$P_X = R_X(0) = 26.25.$$

Fourier transforming,

$$S_X(f) = 25\delta(f) + 0.625[\delta(f-1) + \delta(f+1)].$$

Integrating, you again get

$$P_X = 25 + 0.625 + 0.625 = 26.25.$$

(b) $Y(t)$ is WSS. Specify $R_Y(\tau)$, plot $S_Y(f)$, and then use these results to compute P_Y two different ways.

Solution. The carrier wave's autocorrelation function is $(1/2)\cos(20\pi\tau)$, and therefore

$$R_Y(\tau) = (1/2)R_X(\tau)\cos(20\pi\tau) = (1/2)[25 + (1.25)\cos(2\pi\tau)]\cos(20\pi\tau).$$

This gives

$$P_Y = R_Y(0) = (1/2)[26.25] = 13.125.$$

Fourier transforming,

$$S_Y(f) = (1/4)[S_X(f-10) + S_X(f+10)],$$

which simplifies to

$$S_Y(f) = (5/32)[\delta(f-11)+\delta(f+11)+\delta(f-9)+\delta(f+10)]+6.25[\delta(f-10)+\delta(f+10)]$$

Integrating,

$$P_Y = (5/32)(4) + 6.25(2) = 13.125.$$