6.431 Fall 2002 Quiz 2 Solutions

DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

- You have 50 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has 3 problems that are not necessarily in order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one double-sided, handwritten, 8.5x11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Write your name and your TA's name on the front of the booklet. (2 points)

Problem 1: (28 points)

Let $X_1, X_2, \ldots, X_n, \ldots$ be IID random variables with mean μ and variance σ^2, and let N be a geometric random variable with parameter p. The random variables $N, X_1, X_2, \ldots, X_n, \ldots$ are mutually independent.

The subject of this problem is the study of the first and second order statistics of the random variable

$$Y = X_1 X_2 \ldots X_N,$$

the random product of the random variables $\{X_j\}$.

(a) (12 points) Find the mean of Y.

Solution:

$$\begin{aligned}
\mathbf{E}[Y] &= \mathbf{E}\left[\mathbf{E}[Y|N]\right] = \mathbf{E}\left[\mathbf{E}[X_1 \ldots X_N|N]\right] \\
&= \mathbf{E}\left[\mathbf{E}[X_1|N] \ldots \mathbf{E}[X_N|N]\right] = \mathbf{E}\left[\mu^N\right] \\
&= M_N(s)|_{e^s \leftarrow \mu} = \frac{p\mu}{1-(1-p)\mu}
\end{aligned}$$

(b) (16 points) Find the variance of Y.

Solution:

$$\begin{aligned}
\mathbf{var}(Y) &= \mathbf{E}[Y^2] - \mathbf{E}[Y]^2 = \mathbf{E}\left[\mathbf{E}[Y^2|N]\right] - \mathbf{E}[Y]^2 \\
&= \mathbf{E}\left[\mathbf{E}[X_1^2|N] \ldots \mathbf{E}[X_N^2|N]\right] - \mathbf{E}[Y]^2 \\
&= \mathbf{E}\left[(\sigma^2+\mu^2)^N\right] - \mathbf{E}[Y]^2 \\
&= M_N(s)|_{e^s \leftarrow (\sigma^2+\mu^2)} - M_N^2(s)|_{e^s \leftarrow \mu} \\
&= \frac{p(\sigma^2+\mu^2)}{1-(1-p)(\sigma^2+\mu^2)} - \left(\frac{p\mu}{1-(1-p)\mu}\right)^2
\end{aligned}$$

Problem 2: (62 points)

A wireless communication channel is often modeled as follows: the received random variable Y is related to the transmitted random variable X in the following manner:

$$Y = RX,$$

where R is a random variable independent of X representing a multiplicative noise term. Furthermore, R is assumed to be Rayleigh distributed i.e.

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0.$$

(a) (6 points) If X is a continuous R.V. with PDF $f_X(x)$ and transform $M_X(s)$, what is the PDF and transform of $Z = (-X)$?

Solution:

$$\begin{aligned} M_Z(s) &= M_X(-s) \\ f_Z(z) &= f_X(-z) \end{aligned}$$

(b) (12 points) Find the PDF then transform of $U = R^2$.

Solution: For a non-negative u

$$\begin{aligned} \mathbf{P}(U \leq u) &= \mathbf{P}(R^2 \leq u) = \mathbf{P}(R \leq \sqrt{u}) = \int_0^{\sqrt{u}} f_R(r)dr \\ f_U(u) &= \frac{d}{du}\mathbf{P}(U \leq u) = \frac{1}{2\sqrt{u}} f_R(\sqrt{u}) = \frac{1}{2\sigma^2} e^{-\frac{u}{2\sigma^2}} \\ M_U(s) &= \frac{1/2\sigma^2}{1/2\sigma^2 - s} = \frac{1}{1 - 2\sigma^2 s} \end{aligned}$$

For the remainder of this problem, we assume X to be a standard normal R.V. (i.e. X is Gaussian with mean zero and unit variance.)

(c) (25 points) Find the transform then PDF of Y.

Solution:

$$\begin{aligned} M_Y(s) &= \mathbf{E}\left[e^{sRX}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{sRX}|R\right]\right] \\ &= \mathbf{E}\left[M_X(sR)\right] = \mathbf{E}\left[e^{s^2R^2/2}\right] \\ &= \mathbf{E}\left[e^{s^2U/2}\right] = M_U(s^2/2) \\ &= \frac{1/\sigma^2}{1/\sigma^2 - s^2} = \frac{1}{2}\left(\frac{1/\sigma}{1/\sigma - s}\right) + \frac{1}{2}\left(\frac{1/\sigma}{1/\sigma + s}\right) \\ f_Y(y) &= \begin{cases} 0.5\frac{1}{\sigma}e^{-y/\sigma}, & y \geq 0 \\ 0.5\frac{1}{\sigma}e^{y/\sigma}, & y \leq 0 \end{cases} \\ &= \frac{1}{\sigma}e^{-|y|/\sigma} \end{aligned}$$

(d) (12 points) The receiver observes Y and would like to estimate the transmitted message X. Find the linear least squares estimate $\hat{X}_L$ of X given Y.

Solution:

$$\begin{aligned}
\hat{X}_L &= \frac{\mathbf{cov}(X,Y)}{\mathbf{var}(Y)}Y = \frac{\mathbf{E}[RX^2]}{\mathbf{E}[R^2X^2]}Y \\
&= \frac{\mathbf{E}[R]\mathbf{E}[X^2]}{\mathbf{E}[R^2]\mathbf{E}[X^2]}Y = \frac{1/2\sigma^2\sqrt{2\pi\sigma^2}\sigma^2}{\mathbf{E}[U]}Y \\
&= \frac{1/2\sqrt{2\pi\sigma^2}}{2\sigma^2}Y \\
&= \sqrt{\frac{\pi}{8\sigma^2}}Y
\end{aligned}$$

(e) (7 points) Your friend proposes to use the following non-linear estimator of X:

$$\hat{X} = |Y|.$$

Which estimate is better, $\hat{X}$ or $\hat{X}_L$?

Solution: Look at the average squared error

$$\begin{aligned}
\mathbf{E}[(X-|Y|)^2] &= 1/2.\mathbf{E}[(X-Y)^2|Y \geq 0] + 1/2.\mathbf{E}[(X+Y)^2|Y<0] \\
&= \mathbf{E}[(X-Y)^2] + 1/2\mathbf{E}[(X+Y)^2 - (X-Y)^2|Y<0] \\
&= \mathbf{E}[(X-Y)^2] + 1/2\mathbf{E}[2X.2Y|Y<0] \\
&\geq \mathbf{E}[(X-Y)^2] \\
&\geq \mathbf{E}[(X-\hat{X}_L)^2]
\end{aligned}$$

where we used the fact that X and Y are always of the same sign (including when Y is negative), and the linear least squares estimator minimizes the average squared error of any linear estimator (like Y in the above expression).

Problem 3: (8 points)

Is a random sum of Jointly Gaussian random variables Gaussian ? Explain.

Solution: Not true in general: see problem set!