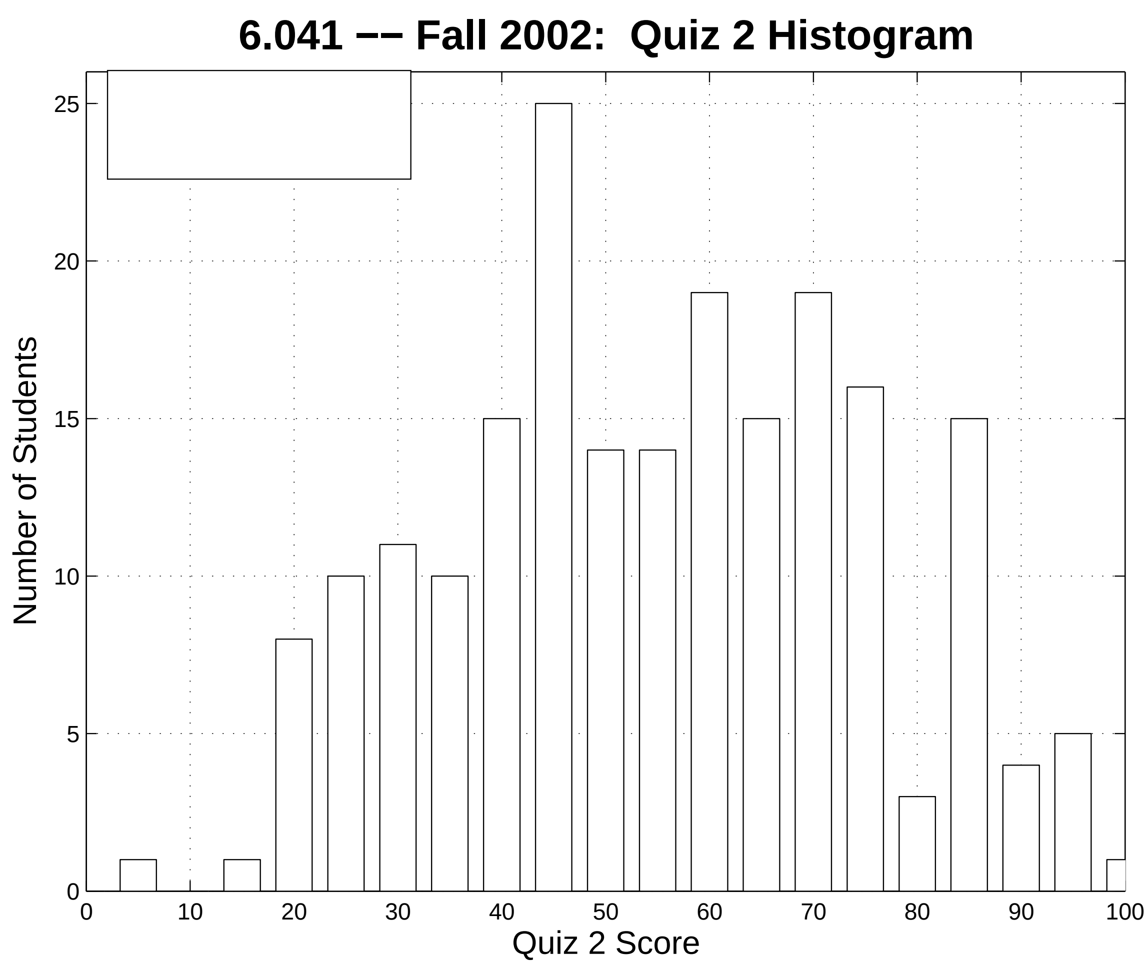
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Quiz 2 Results (6.041 only)

(6.431 results will be available before Wed)

- Solutions to the 6.041 quiz are on the next page.
- *Regrade Policy:* Students who feel there is an error in the grading of their quiz have until **Wednesday, November 13** to submit the regrade request to their TA. **Do not write anything at all on the exam booklet!** Instead attach a note on a separate piece of paper explaining the putative error. Any attempt to modify a quiz booklet is considered a serious breach of academic honesty. We photocopy a substantial fraction of the quizzes before they are returned, implying there exists a nonzero probability of us catching such a change. We also reserve the right to regrade the entire quiz, not just the problem with the putative error.



6.041 Quiz 2 Solutions

Problem 1: The key insight is that Y_m is a sum of independent (perhaps scaled) random variables.

(a) (14 pts) Because the V_k's are identically distributed, $\mathbf{E}[V_k] = \mathbf{E}[V_1] = \mu_V$. By linearity of expectation and the fact that X is zero-mean,

$$\mathbf{E}[Y_m] = \mathbf{E}[\alpha^m X + \sum_{i=1}^{m} \alpha^{m-i} V_i] = \alpha^m \mathbf{E}[X] + \sum_{i=1}^{m} \alpha^{m-i} \mathbf{E}[V_i] = \sum_{i=1}^{m} \alpha^{m-i} \mu_V \quad \left[= \frac{\mu_V (1-\alpha^m)}{1-\alpha}\right]$$

(b) (14 pts) All V_k's have the identical transform $M_V(s)$ and $M_X(s) = e^{\sigma_X^2 s^2/2}$. By properties of the exponential and the stated independence of X and the V_k's,

$$\begin{aligned}
\mathbf{E}[e^{sY_m}] &= \mathbf{E}[e^{s\left(\alpha^m X + \sum_{i=1}^{m} \alpha^{m-i} V_i\right)}] = \mathbf{E}[e^{s\alpha^m X} \prod_{i=1}^{m} e^{s\alpha^{m-i} V_i}] = \mathbf{E}[e^{s\alpha^m X}] \prod_{i=1}^{m} \mathbf{E}[e^{s\alpha^{m-i} V_i}] \\
&= M_X(\alpha^m s) \prod_{i=1}^{m} M_V(\alpha^{m-i} s) = e^{\sigma_X^2 \alpha^{2m} s^2/2} \prod_{i=1}^{m} M_V(\alpha^{m-i} s)
\end{aligned}$$

(c) (14 pts) Conditioning on the possible values of N,

$$\begin{aligned}
\mathbf{E}[e^{sY_N}] &= \mathbf{E}[e^{sY_N} \mid N = m] p_N(m) + \mathbf{E}[e^{sY_N} \mid N = m+1] p_N(m+1) = \mathbf{E}[e^{sY_m}] \frac{3}{4} + \mathbf{E}[e^{sY_{m+1}}] \frac{1}{4} \\
&= \frac{3}{4} \left(e^{\sigma_X^2 \alpha^{2m} s^2/2} \prod_{i=1}^{m} M_V(\alpha^{m-i} s) \right) + \frac{1}{4} \left(e^{\sigma_X^2 \alpha^{2(m+1)} s^2/2} \prod_{i=1}^{m+1} M_V(\alpha^{m+1-i} s) \right)
\end{aligned}$$

Problem 2: The key insight is that, for every $k \geq 1$, Y_k is independent of each V_j for $j > k$. Also, given the V_k's are zero-mean (as well as X) implies the Y_k's are zero mean.

(a) (14 pts) Given $\text{var}(Y_k) = \text{var}(Y_0) = \text{var}(X) = \sigma_X^2$ and the independence between V_k and Y_{k-1},

$$\text{var}(Y_k) = \text{var}(\alpha Y_{k-1} + V_k) = \alpha^2 \text{var}(Y_{k-1}) + \text{var}(V_k) \quad \Rightarrow \quad \text{var}(V_k) = \sigma_X^2 - \alpha^2 \sigma_X^2 = \left(1 - \alpha^2\right) \sigma_X^2$$

(b) (14 pts) We know a Gaussian multiplied by a constant remains Gaussian and also that the sum of independent Gaussians is a Gaussian. Hence, with $Y_0 = X$ given to be Gaussian, Y_k for $k \geq 1$ will also be Gaussian provided the V_k's are Gaussian; use mean as given and variance as found in part (a).

(c) (14 pts) Using the recursion, $Y_{i-1} = \alpha Y_{i-2} + V_{i-1}$ and $Y_i = \alpha Y_{i-1} + V_i = \alpha^2 Y_{i-2} + \alpha V_{i-1} + V_i$. Therefore, by linearity of expectation, exploiting independence and employing the "Pull-Through Property" (proved in problem set 7),

$$\begin{aligned}
\mathbf{E}[Y_i Y_{i-1} | Y_{i-2}] &= \mathbf{E}[(\alpha^2 Y_{i-2} + \alpha V_{i-1} + V_i)(\alpha Y_{i-2} + V_{i-1}) | Y_{i-2}] \\
&= \alpha^3 \mathbf{E}[Y_{i-2}^2 | Y_{i-2}] + \alpha^2 \mathbf{E}[Y_{i-2} V_{i-1} | Y_{i-2}] + \alpha^2 \mathbf{E}[V_{i-1} Y_{i-2} | Y_{i-2}] + \alpha \mathbf{E}[V_{i-1}^2 | Y_{i-2}] + \\
&\quad \alpha \mathbf{E}[V_i Y_{i-2} | Y_{i-2}] + \mathbf{E}[V_i V_{i-1} | Y_{i-2}] \\
&= \alpha^3 Y_{i-2}^2 + 2\alpha^2 Y_{i-2} \mathbf{E}[V_{i-1}] + \alpha \mathbf{E}[V_{i-2}^2] + \alpha Y_{i-2} \mathbf{E}[V_i] + \mathbf{E}[V_i] \mathbf{E}[V_{i-1}] \\
&= \alpha^3 Y_{i-2}^2 + \alpha \left(1 - \alpha^2\right) \sigma_X^2
\end{aligned}$$

where the last step follows from the V_k's being zero-mean and so $\mathbf{E}[V_i^2] = \text{var}(V_i)$.

(d) (14 pts) We know the Y_k's are zero-mean and $\text{var}(Y_k) = \sigma_X^2$. Thus,

$$g_L(Y_k) = \mathbf{E}[Y_{k-1}] + \frac{\text{cov}(Y_{k-1} Y_k)}{\text{var}(Y_k)} (Y_k - \mathbf{E}[Y_k]) = \frac{\text{cov}(Y_k Y_{k-1})}{\sigma_X^2} Y_k = \alpha Y_k$$

because

$$\text{cov}(Y_k Y_{k-1}) = \mathbf{E}[Y_k Y_{k-1}] - \mathbf{E}[Y_k] \mathbf{E}[Y_{k-1}] = \mathbf{E}[(\alpha Y_{k-1} + V_k) Y_{k-1}] = \mathbf{E}[\alpha Y_{k-1}^2 + V_k Y_{k-1}] = \alpha \sigma_X^2$$

(The same answer could be obtained via the Law of Iterated Expectations on the answer for part (c).)