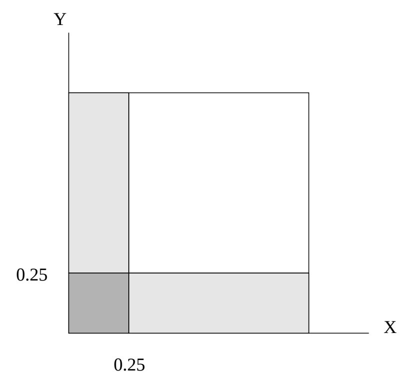
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Quiz 2 Review Solutions

1. (a) The joint PDF of X and Y, being independent, is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} 1 & 0 < x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The probability that a message is received 15 minutes after A sent both messages is illustrated below.



From the graph, we are defining event A as $X \le 0.25$ and event B as $Y \le 0.25$.

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

$$\mathbf{P}(A) = \mathbf{P}(X \le 15) = 0.25$$

$$\mathbf{P}(B) = \mathbf{P}(Y \le 15) = 0.25$$

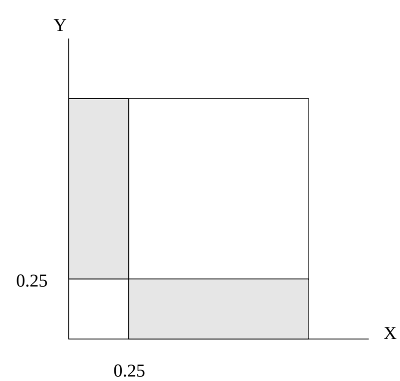
Since event A and B are independent, i.e. receiving of one message does not effect the receiving of the other one,

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B) = 0.25^2 = 0.0625$$

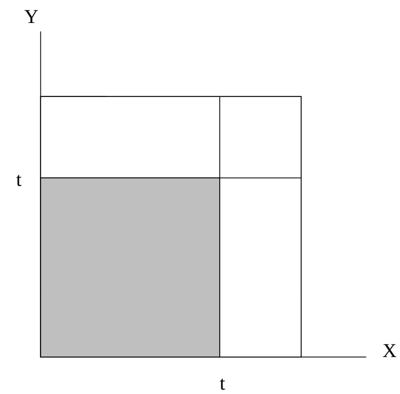
$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 0.25 + 0.25 - 0.0625 = 0.4375$$

(b) Let B = (message received but not verified within 15 minutes)

$$\mathbf{P}(B) = \mathbf{P}(X \le 0.25 \cap Y > 0.25) + \mathbf{P}(X > .25 \cap Y \le .25) = 0.1875 + 0.1875 = 0.3750$$



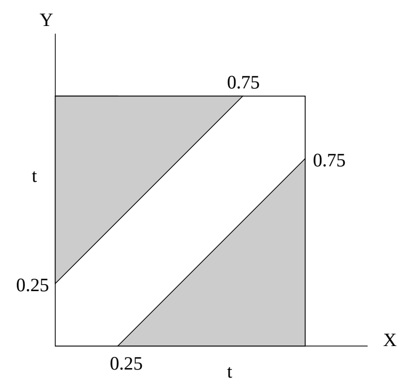
(c) $\mathbf{P}(T \le t) = \mathbf{P}(X \le t \cap Y \le t) = \mathbf{P}(X \le t)\mathbf{P}(Y \le t) = t \cdot t = t^2$



From this we can deduce the CDF that by differentiation yields the PDF:

$$F_T(t) = \begin{cases} 0 & -\infty < t \le 0 \\ t^2 & 0 < t \le 1 \\ 1 & 1 < t < \infty \end{cases} \quad \Rightarrow \quad f_T(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) The event that the clerk will be there to receive the message is precisely $\mathbf{P}(|X - Y| > 0.25)$. We can deduce the exact probability by analyzing it graphically.



$$\mathbf{P}(|X - Y| > 0.25) = 2 \cdot (0.5 \cdot 0.75^2) = 0.5625$$

(e) We know the strategy from (d) has 0.5625 probability of verification. Also we know that $\mathbf{P}(T \le 0.75) = 0.75^2 = 0.5625$. Therefore, it is same whether we use strategy proposed in part (d) or let him go after 45 minutes.

2. First let's write out the properties of all of our random variables. Let us also define K to be the number of members attending a meeting, and B to be the Bernoulli random variable describing whether or not a member attends a meeting.

$$\begin{aligned} M_N(s) &= \frac{(1-p)e^s}{1-pe^s}, \quad \mathbf{E}[N] = \frac{1}{1-p}, \quad \text{var}(N) = \frac{p}{(1-p)^2} \\ M_M(s) &= \frac{\lambda}{\lambda - s}, \quad \mathbf{E}[M] = \frac{1}{\lambda}, \quad \text{var}(M) = \frac{1}{\lambda^2} \\ M_B(s) &= 1 - q - qe^s, \quad \mathbf{E}[B] = q, \quad \text{var}(B) = q(1-q) \end{aligned}$$

(a) Since B and N are independent,

$$\mathbf{E}[K] \;=\; \mathbf{E}[N] \cdot \mathbf{E}[B] = \frac{q}{1-p}, \quad \text{var}(K) = \mathbf{E}[N] \cdot {\sigma_B}^2 + {\mu_B}^2 \cdot \text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}$$

(b) We begin first by finding $M_K(s)$ and then we evaluate $p_K(1)$ using properties of the transform. To do this, we recognize that $K = B_1 + B_2 + B_3 + ... + B_N$.

$$\begin{aligned}
M_K(s) &= M_N(s)\Big|_{e^s = M_B(s)} = \frac{(1-p)(1-q+qe^s)}{1-p(1-q+qe^s)} \\
p_K(1) &= \frac{d}{de^s}M_K(s)\Big|_{e^s=0} = \left[\frac{(1-p)q}{1-p(1-q+qe^s)} + \frac{pq(1-p)(1-q+qe^s)}{[1-p(1-q+qe^s)]^2}\right]\Bigg|_{e^s=0} \\
&= \frac{(1-p)q}{1-p(1-q)} + \frac{pq(1-p)(1-q)}{[1-p(1-q)]^2}
\end{aligned}$$

(c) Let G be the total money brought to the meeting. Then $G = M_1 + M_2 + M_3 + ... + M_K$. Thus we can write the transform of G as follows:

$$M_G(s) = M_K(s)\Big|_{e^s = M_M(s)} = \frac{(1-p)\left[1-q+q\left(\frac{\lambda}{\lambda-s}\right)\right]}{1-p\left[1-q+q\left(\frac{\lambda}{\lambda-s}\right)\right]}$$

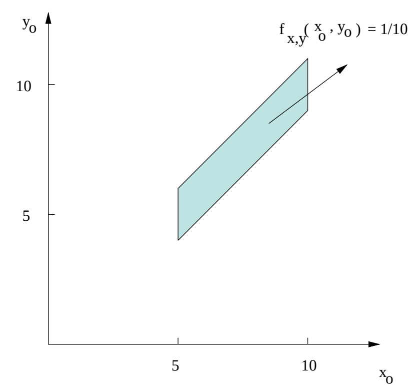
3. (a) The minimum mean squared error estimator $g(Y)$ is known to be $g(Y) = \mathbf{E}[X|Y]$. Let us first find $f_{X,Y}(x,y)$. Since $Y = X + W$, we can write

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2} & x-1 \leq y \leq x+1 \\ 0 & \text{otherwise} \end{cases}$$

and, therefore,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \frac{1}{10} & x-1 \leq y \leq x+1 \text{ and } 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

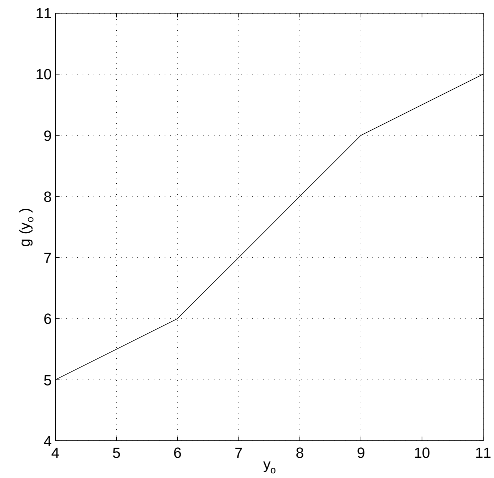
as shown in the plot below.



We now compute $\mathbf{E}[X|Y]$ by first determining $f_{X|Y}(x|y)$. This can be done by looking at the horizontal line crossing the compound PDF. Since $f_{X,Y}(x,y)$ is uniformly distributed in the defined region, $f_{X|Y}(x|y)$ is uniformly distributed as well. Therefore,

$$g(y) = \mathbf{E}[X|Y=y] = \begin{cases} \frac{5+(y+1)}{2} & 4 \leq y < 6 \\ y & 6 \leq y \leq 9 \\ \frac{10+(y-1)}{2} & 9 < y \leq 11. \end{cases}$$

The plot of $g(y)$ is shown here.



(b) The linear least squares estimator has the form

$$g_L(Y) = \mathbf{E}[X] + \frac{\text{cov}(X,Y)}{\sigma_Y^2}(Y - \mathbf{E}[Y])$$

where $\text{cov}(X,Y) = \mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]$. We compute $\mathbf{E}[X] = 7.5$, $\mathbf{E}[Y] = \mathbf{E}[X]+\mathbf{E}[W] = 7.5$, $\sigma_X^2 = (10-5)^2/12 = 25/12$, $\sigma_W^2 = (1-(-1))^2/12 = 4/12$ and, using the fact that X and W are independent, $\sigma_Y^2 = \sigma_X^2 + \sigma_W^2 = 29/12$. Furthermore,

$$\begin{aligned}
\text{cov}(X,Y) &= \mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])] = \mathbf{E}[(X-\mathbf{E}[X])(X-\mathbf{E}[X]+W-\mathbf{E}[W])] \\
&= \mathbf{E}[(X-\mathbf{E}[X])(X-\mathbf{E}[X])] + \mathbf{E}[(X-\mathbf{E}[X])(W-\mathbf{E}[W])] \\
&= \sigma_X^2 + \mathbf{E}[(X-\mathbf{E}[X])]\mathbf{E}[(W-\mathbf{E}[W])] = \sigma_X^2 = 25/12.
\end{aligned}$$

Note that we use the fact that $(X-\mathbf{E}[X])$ and $(W-\mathbf{E}[W])$ are independent and $\mathbf{E}[(X-\mathbf{E}[X])] = 0 = \mathbf{E}[(W-\mathbf{E}[W])]$. Therefore,

$$g_L(Y) = 7.5 + \frac{25}{29}(Y - 7.5).$$

The linear estimator $g_L(Y)$ is compared with $g(Y)$ in the following figure. Note that $g(Y)$ is piecewise linear in this problem.