LECTURE 12:

A bit more on continuous R.V.'s, and derived distributions

Readings:

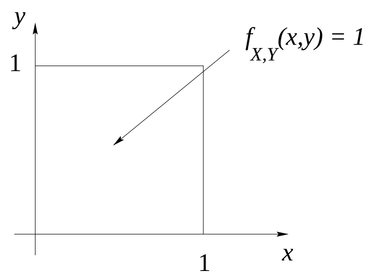
- Review Chapter 3
- Section 4.2

Lecture Outline:

- General formula for strictly monotonic fct
- Distribution of sum of independent R.V.'s
- Sum of independent Normal R.V.'s
- Continuous Bayes' rule

Function of more than one R.V.

Example: X and Y are two continuous R.V.'s with the following joint PDF:



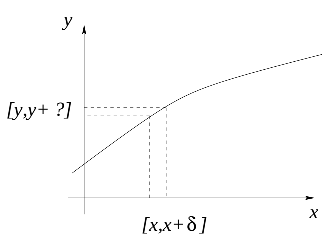
Find the PDF of $Z = g(X, Y) = Y/X$

$F_Z(z) =$ $\qquad z \leq 1$

$F_Z(z) =$ $\qquad z \geq 1$

A general formula

Question: Distribution of $Y = g(X)$?
g is strictly monotonic



- Events

$$\{x \leq X \leq x + \delta\}$$
$$= \{g(x) \leq Y \leq g(x + \delta)\}$$
$$\approx \{g(x) \leq Y \leq g(x) + \delta|(dg/dx)(x)|\}$$

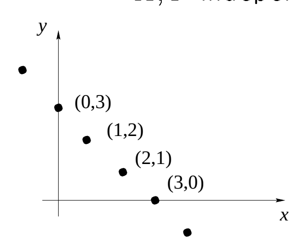
- Answer:

$$f_X(x) = f_Y(y)\left|\frac{dg}{dx}(x)\right|$$

where $y = g(x)$

The distribution of $X + Y$

The discrete case: $W = X + Y$,
X, Y independent



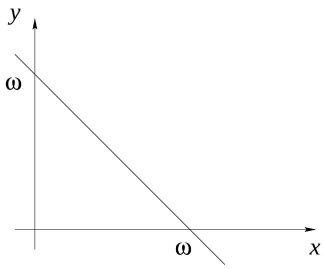
$$p_W(w) = \mathrm{P}(X + Y = w)$$
$$= \sum_x \mathrm{P}(X = x)\mathrm{P}(Y = w - x)$$
$$= \sum_x p_X(x)p_Y(w - x)$$

Mechanics:

- Put the pmf's on top of each other
- Flip the pmf of Y
- Shift the flipped pmf by w (to the right if $w > 0$)
- Cross-multiply and add

The continuous case: $W = X + Y$, X, Y independent

- $f_{X,Y}(x,y) = f_X(x) f_Y(y)$



- $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$

Mechanics:

- Put the pdf's on top of each other
- Flip the pdf of Y
- Shift the flipped pdf by w (to the right if $w > 0$)
- Cross-multiply and integrate

Two independent normal R.V.'s

$X \sim N(\mu_x, \sigma_x^2), Y \sim N(\mu_y, \sigma_y^2)$, independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\}$$

- PDF is constant on the ellipse where
$$\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2}$$
is constant

- Ellipse is a circle when $\sigma_x = \sigma_y$

The sum of independent normal R.V.'s

$X \sim N(0, \sigma_x^2)$, $Y \sim N(0, \sigma_y^2)$, independent

Question: Distribution of $W = X + Y$?

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-(w-x)^2/2\sigma_y^2}\,dx \\ \text{(algebra)} &= c e^{-\gamma w^2} \end{aligned}$$

Answer: W is normal

- mean=
- variance=

Continuous Bayes rule

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y \mid x)}{f_Y(y)}$$

Common case: $Y = X + N$

- signal X, with additive noise N
- N independent from X

Then, $f_{Y|X}(y \mid x) = f_N(y - x)$

Remarkable fact:

if X and N are normal,
then $f_{X|Y}(x \mid y)$ is a normal PDF,
for any given y.