THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:541 **Stochastic Signals and Systems** **Fall 2007**

Quiz I

There are 3 questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

Useful facts:

$\sum_{k=0}^{K} z^k = \frac{1-z^{K+1}}{1-z}$ $(z \neq 1)$	$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$	$E[X] = \int_0^\infty (1 - F_X(x))dx$ X non-negative

1. *(30 points)* Rutgera Univera, the world famous Rutgers University ECE graduate student has opened a painting business called High Probability Coverage (HPC). Her workers are armed with paintball guns which they shoot at the object to be painted. Trying to cut costs, Rutgera bought used paintball guns and their triggers are a bit unpredictable. If you hold the trigger down, paint balls are shot, but the times T_i in seconds between shots i and $i+1$ are mutually independent exponential random variable with mean $1/\lambda$. That is $f_{T_i}(t_i) = \lambda e^{-\lambda t_i}$.

   Rutgera has trained her workers to paint in straight lines by sweeping the muzzle of the gun at a constant velocity v in distance-units per second. When a paint ball hits the target, it paints a unit-radius splotch.

   (a) *(10 points)* What is the distribution on the distance L_i between the centers of paint ball splotches i and $i+1$?

   SOLUTION: *The distance between the centers of successive splotches is* $L_i = vT_i$ *where* v *is a constant. Thus,* $f_L(\ell) = (\lambda/v)e^{-\ell\lambda/v}$.

   (b) *(10 points)* What is the probability that any given two successive splotches will not overlap?

   SOLUTION: *Splotches will not overlap if the distance between their centers exceeds two units. The probability any two splotches exceed distance* 2 *units is then*

   $$\int_2^\infty f_L(\ell)d\ell = e^{-2\lambda/v}$$

   (c) *(10 points)* What is the probability that a sequence of K splotches will overlap but the $(K+1)$st splotch will leave a gap?

   SOLUTION: *The "stopping probability" (where a gap occurs) is* $p = e^{-2\lambda/v}$*. So the "run length" (number of overlaps before a gap) distribution is geometric* $p_K(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$*. A single splotch is assumed to overlap with itself.*

2. *(35 points)* We glossed over the mechanics of the central limit theorem in class. You're going to explore them here.

   (a) *(5 points)* The moment generating function of a random variable X is

   $$\Phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

   It's three term Taylor series about $s = 0$ is $\Phi_X(s) \approx a + bs + c\frac{s^2}{2}$. Assuming X is zero mean and has variance σ^2, please determine a, b, and c.
   SOLUTION:

   $$\Phi_X(s) \approx \Phi_X(0) + \Phi_X'(0)s + \Phi_X''(0)s^2/2 = 1 + E[X]s + E[X^2]s^2/2 = 1 + \sigma^2\frac{s^2}{2}$$

   so $a = 1$, $b = 0$ *and* $c = \sigma^2$.

   (b) *(10 points)* We form the random variable $W_n = \frac{1}{\sqrt{n\sigma^2}}\sum_{i=1}^{n} X_i$ where the X_i are mutually independent and each has distribution $f_X(x)$. What is $E[W_n]$? What is $E[W_n^2]$? Write down an exact expression for $\Phi_{W_n}()$ in terms of $\Phi_X()$. BE CAREFUL.
   SOLUTION: *Un-normalized, the MGF of* $\sum_{i=1}^{n} X_i$ *would be* $(\Phi_X(s))^n$. *Normalized, the MGF of each of the* $X_i/\sqrt{n\sigma^2}$ *becomes* $\Phi_X(s/\sqrt{n\sigma^2})$ *so that*

   $$\Phi_{W_n}(s) = \left(\Phi_X\left(\frac{s}{\sqrt{n\sigma^2}}\right)\right)^n$$

   (c) *(5 points)* Write down an approximation for $\Phi_{W_n}(s)$ in terms of the Taylor approximation for $\Phi_X(s)$.
   SOLUTION:

   $$\Phi_{W_n}(s) \approx \left(1 + \frac{\sigma^2}{2}\frac{s^2}{n\sigma^2}\right)^n = \left(1 + \frac{s^2}{2n}\right)^n$$

   (d) *(10 points)* Show that $\lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$.
   SOLUTION:

   $$(1+v)^n = \sum_{k=0}^{n}\binom{n}{k}v^k = \sum_{k=0}^{n}\frac{n!}{(n-k)!}\frac{v^k}{k!}$$

   Letting $v = z/n$ *we have*

   $$(1+z/n)^n = \sum_{k=0}^{n}\frac{n!}{n^k(n-k)!}\frac{z^k}{k!}$$

   We then note that for any given value of k in the sum, for large enough n we have

   $$\frac{n!}{n^k(n-k)!} \approx 1$$

   Furthermore, for finite z, the terms in the sum diminish rapidly (geometrically) for n large enough that $z/n < 1$. *So only terms* $k \ll n$ *are important in the sum. Thus,*

   $$\lim_{n\to\infty}(1+z/n)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z$$

(e) *(5 points)* What is the limit as $n \to \infty$ for the Taylor approximation to $\Phi_{W_n}(s)$?
SOLUTION:

$$\lim_{n\to\infty} \Phi_{W_n}(s) = e^{s^2/2}$$

We've shown that the moment generating function (also called characteristic function) converges to a Gaussian MGF. However, that does not necessarily mean that the DISTRIBUTIONS CONVERGE. This is a WHOLE LOT more difficult to prove. It also depends on the type of convergence you're after – for instance, is a Gaussian distribution still a Gaussian distribution even if you take a single point and set the value to zero? Clearly it's still a probability distribution because the integral is still unity. These sorts of nitpicking details are what bedevil you. The upshot is that if we assume we're looking only at continuous distributions and continuous $\Phi_X(s)$, we end up being ok.

But in the FULL Central Limit Theorem, you have to cover ALL cases including sums of correlated *random variables, non-identically distributed random variables and the like. So it's even more involved. However, the convergence is so fast that the CLT is a very robust result –so long as we can be assured that the variables aren't perfectly correlated and $\sum_{k=0}^{\infty}\sigma_k^2$ approaches infinity. Therefore, when teaching it for use (as opposed to for an analysis course), we skirt the details, wave our hands, and supply some motivation that it's true as in this problem.*

And then we forget about it.

3. *(35 points)* Every morning, a large family of very sophisticated wombats chooses D from a geometric distribution $p_D(d) = p(1-p)^{d-1}$, $d = 1,2,\ldots$. D is the number wombats who will make that morning's perilous journey across a busy road to bring back food. Cars come down the road with an average frequency of λ cars per second. The interarrival time Δ_i between successive cars is an exponential random variable with mean $1/\lambda$. That is $f_\Delta(\delta) = \lambda e^{-\lambda\delta}$. The Δ_i are mutually independent.

The time T it takes any given wombat to cross the road is also an exponential random variable but with different mean $1/\kappa$. The T_i are assumed mutually independent. In addition, it is assumed that the $\{T_i\}$ and $\{\Delta_i\}$ are also mutually independent.

Any wombats in the road when a car comes along are instantly killed.

(a) *(10 points)* Show that $\Phi_T(s) = \frac{\kappa}{\kappa-s}$ and that $\Phi_D(s) = \frac{pe^s}{1-(1-p)e^s}$.
SOLUTION:

$$\Phi_T(s) = \int_0^\infty e^{st}\kappa e^-\kappa t dt = \int_0^\infty \kappa e^-(\kappa-s)tdt = -\frac{\kappa}{\kappa-s}e^-(\kappa-s)t|_0^\infty = \frac{\kappa}{\kappa-s}$$

$$\Phi_D(s) = \sum_{d=1}^{\infty} p(1-p)^d e^{sd} = pe^s \sum_{d=1}^{\infty} [(1-p)e^s]^{d-1} = \frac{pe^s}{1-(1-p)e^s}$$

(b) *(10 points)* Let the random variable W be the time required for all the wombats to get across the road. Write down an analytic expression for the probability that all wombats survive in terms of $f_W(w)$ and $f_\Delta(\delta)$. Simplify your result in terms of $\Phi_W()$.
SOLUTION: *A wombat gets run over if $W > \Delta$. Since Δ and W are independent*

$$Prob[W \le \Delta] = \int_0^\infty f_W(w)\int_w^\infty f_\Delta(\delta)d\delta dw = \int_0^\infty f_W(w)e^{-\lambda w}dw = \Phi_W(-\lambda)$$

(c) *(10 points)* Suppose the wombats cross sequentially (the next one starts just after the one ahead reaches the other side). What is the probability that they all survive in terms of p, κ and λ?

SOLUTION: *W is a random sum of iid random variables so*

$$\Phi_W(s) = \Phi_D(\ln \Phi_T(s)) = \frac{p\kappa}{p\kappa - s}$$

The survival probability is

$$Prob[W \le \Delta] = \Phi_W(-\lambda) = \frac{p\kappa}{p\kappa + \lambda}$$

(d) *(5 points)* What value of p maximizes the probability that no wombats are killed?

SOLUTION: *Intuitively, having $p = 1$ makes the most sense – send only one wombat across. Sending sequential wombats only increases the number of chances the wombats have to get hit by a car. Analytically, the expression*

$$Prob[W \le \Delta] = \frac{p\kappa}{p\kappa + \lambda}$$

is monotone increasing in p. Thus, $p = 1$ (sending exactly one wombat) maximizes the probability of survival.