UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 3
Spring 2007

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Problem 3.1

a) The probability that one die lands on a six is $\frac{1}{6}$. Since the outcome of the one die is independent of the other, the probability that at least one lands on a six is:

$$\frac{1}{6}+\frac{1}{6}-\frac{1}{36}=\frac{11}{36}$$

because of the relation:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B).$$

b) Given that the two dice land on different numbers, the outcome of one die is no longer independent of the outcome of the other die. In fact, our event space changes. By enumeration we can see that there are 10 "favorable" outcomes, i.e. there are 10 possible rolls where one of the dice being a six, and 30 possible rolls where the dice land on different numbers. Since the likelihood of any of the 30 rolls is equal we conclude that the conditional probability we are after is: $\frac{1}{3}$.

Problem 3.2

Through the first n coins, Alice and Bob are equally likely to have flipped the same number of heads as the other (since they are using fair coins and each flip is independent of the other flips). Given this, when Bob flips his last coin but Alice doesn't flip a coin, Bob has a 1/2 chance of getting a head and thus having more heads than Alice. Formally:

Let A, B be the number of heads that Alice and Bob get in n coins, respectively. Let also X be the number of head (0 or 1) in the (n+1)-st coin of Bob. We want $P(A<B+X)$. Now, $P(A<B+X)=P(A=B\cap X=1)+P(A<B)$. Also, $1=P(A=B\cap X=1)+P(A=B\cap X=0)+P(A<B)+P(A>B)$. Now, $P(A=B\cap X=1)=P(A=B\cap X=0)$ and $P(A<B)=P(A>B)$. It follows that $P(A<B+X)=1/2$.

Problem 3.3

Since $P(C|A)=1-P(B)$, $P(A)=k\cdot P(B)$, and $P(A\cap B)=P(A)\cdot P(B)$ by independence,

$$P(A|C)=\frac{P(C|A)\cdot P(A)}{P(C)}=\frac{(1-P(B))\cdot P(A)}{P(A)+P(B)-2\cdot P(A\cap B)}=\frac{k\cdot P(B)(1-P(B))}{(k+1)P(B)-2\cdot k\cdot P(B)^2}$$

$$=\frac{k-P(A)}{k+1-2\cdot P(A)}$$

Since

$$k \geq 1 > P(A) > 0$$

$$P(A|C) \geq \frac{k}{k+1}$$

Problem 3.4

Let event A be the event that the student is not overly stressed, and let A^c be the event that the student is in fact overly stressed. Note that these two events partition the sample space. Now let event B be the event that the student's test result indicates that s/he is not overly stressed. The desired probability, $P(A \mid B)$, is found by Bayes' rule:

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)} = \frac{0.005 \cdot 0.95}{0.005 \cdot 0.95 + 0.995 \cdot 0.15} \approx \boxed{0.0308}.$$

Problem 3.5

This is a straightforward application of Bayes's Theorem:

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$$

Let A be the event that you read in a 1, and let event B_1 be the event that there was in fact a 1, and B_0 be the event that there was in fact a 0. The result follows immediately from this, yielding:

$$P(B_1|A) = .895$$

Problem 3.6

Let T be the bit transmitted, R be the bit received. We know from the problem that,

$$P(R = 0 \mid T = 0) = 1 - \epsilon \qquad P(R = 1 \mid T = 1) = 1 - \delta$$

1. $P(R = 1 \mid T = 0) = 1 - P(R = 0 \mid T = 0) = 1 - (1 - \epsilon) = \epsilon$
   $P(R = 0 \mid T = 1) = 1 - P(R = 1 \mid T = 1) = 1 - (1 - \delta) = \delta$

2. Let $P(T = 0) = p$ and $P(T = 1) = 3p$, we know

   $$P(T = 0) + P(T = 1) = 4p = 1,$$

   Therefore, $P(T = 0) = \frac{1}{4}, P(T = 1) = \frac{3}{4}$.

   $$P(\text{Error}) = P(T = 0)P(R = 1 \mid T = 0) + P(T = 1)P(R = 0 \mid T = 1) = (\frac{1}{4})\epsilon + (\frac{3}{4})\delta$$

3. The first step that the transmitter takes is to transmit a "1" or "0" with the corresponding probability of sending a "1" or "0". Then, the signal is repeated $2n+1$ times. These two actions are independent.

   Let N_0 and N_1 be the number of zeros and ones received in the $2n+1$ bits respectively,

$$\begin{aligned} P(\text{Error}) &= P(T=0 \cap N_0 < N_1) + P(T=1 \cap N_0 > N_1) \\ &= P(T=0)P(N_0 < N_1 \mid T=0) + P(T=1)P(N_0 > N_1 \mid T=1) \end{aligned}$$

$$\begin{aligned} P(N_0 < N_1 \mid T=0) &= \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} (\epsilon)^i (1-\epsilon)^{2n+1-i} \\ P(N_0 > N_1 \mid T=1) &= \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} (\delta)^i (1-\delta)^{2n+1} \end{aligned}$$

   Therefore,

$$P(\text{Error}) = \frac{1}{4} \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} (\epsilon)^i (1-\epsilon)^{2n+1-i} + \frac{3}{4} \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} (\delta)^i (1-\delta)^{2n+1}$$

4. Given that $\epsilon = \delta = \frac{1}{10}$,

$$P(\text{Error}) = \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} (\frac{1}{10})^i (\frac{9}{10})^{2n+1-i}$$

   For $n = 11$, $P(\text{Error}) = 4.68E-07$. Therefore, the transmitter must repeat the bit 23 times in order to guareentee a probability of error less than 10^{-6}.

Problem 3.7

1. No. A and B are not independent. To see this, note that $A \subset B$, hence $P(A \cap B) = P(A)$. This is equal to $P(A) \cdot P(B)$ only when $P(B) = 1$ or $P(A) = 0$. But in our example, clearly $P(B) < 1$ and $P(A) > 0$. Hence $P(A \cap B) \neq P(A)P(B)$, and thus A and B are not independent.

2. Yes. Conditioned on C, A will happen if and only if Imno meets 5 people during the second week. Hence $P(A|C) = 1/5$.

   If Imno made 5 friends in the first week, she is certain to make more than 5 friends in total. Hence $P(B|C) = 1$.

   If A happens, B will also happen, so clearly $P(A \cap B|C) = P(A|C) = P(A|C) \cdot P(B|C)$, therefore A and B are conditionally independent. Note that A and B were not independent prior to the conditioning.

3. No. We found in part (b) that $P(A|C) = 1/5$, whereas $P(A) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$. Hence A and C are not independent. (Note: $P(A|C) = \frac{P(A \cap C)}{P(C)}$ by definition, and independence implies that $P(A|C) = \frac{P(A) \cdot P(C)}{P(C)} = P(A)$. Hence $P(A|C) = P(A)$ is a necessary and sufficient condition for checking indepence, as long as $P(C) > 0$.) $P(B|C) = 1$, as we found above, but clearly $P(B) < 1$, hence B and C are not independent.

4. Let F_i where $(i = 1, ..., 5)$ denote the event that in the first week i friends were made. Similarly let S_i denote the event that in the second week i friends were made. Let T_j where $(j = 2, ..., 10)$ denote the event that the total number of friends made in the two weeks is j.

$$
\begin{aligned}
P(\text{“2 in first''}|\text{“6 total''}) &\overset{def}{=} P(F_2|T_6) \\
&= \frac{P(T_6|F_2) \cdot P(F_2)}{P(T_6)} \\
&= \frac{P(S_4 \cdot) P(F_2)}{\sum_{i=1}^{5} P(F_i \cap S_{(6-i)})} \\
&= \frac{\frac{1}{5} \cdot \frac{1}{5}}{5 \cdot \frac{1}{5}\frac{1}{5}} = \frac{1}{5},
\end{aligned}
$$

where the second equality uses Bayes' Rule, and the third equality uses the Total Probability Theorem, and the last equality uses the fact that the numbers of friends made in each week are independent.

Similarly, $P(F_3|T_6) = 1/5$.

Problem 3.8

The network of friendship is best represented as a square with diagonals, with the corners labelled A (Anne), B (Betty), C (Chloe), and D (Daisy). Each link of the network is absent with probability p, independent of the status of any other link. We write XY for the event that the direct link XY is present, and XY^c when that link is absent. We write $X \leftrightarrow Y$ for the event that person X's rumour is (possibly indirectly) heard by person Y. Also note that all communication in this p[roblem is bidirectional; that is, if person X's rumour can be heard by person Y, then person Y's rumour can also be heard by person X.

1. If the link between A and D is present (event AD), Daisy hears the rumour from Anne directly. If the direct link between A and D is absent (event AD^c, i.e. Anne and Dasiy have quarrelled), Daisy might still hear the rumour depending on the status of the links involving Betty or Chloe. There are $2^6 = 64$ possible network states, implying enumeration of the sample space and summing up outcomes that correspond to $A \leftrightarrow D$ is cumbersome. Rather, we repeatedly apply the Total Probability Theorem and exploit

independence between links:

$$
\begin{aligned}
P(A \leftrightarrow D) &= \underbrace{P(A \leftrightarrow D|AD)}_{1}\,\underbrace{P(AD)}_{1-p} + P(A \leftrightarrow D|AD^c)\,\underbrace{P(AD^c)}_{p} \quad , \\
P(A \leftrightarrow D|AD^c) &= P(A \leftrightarrow D|AD^c \cap BC)\,\overbrace{P(BC|AD^c)}^{1-p} + \\
&\quad P(A \leftrightarrow D|AD^c \cap BC^c)\,\underbrace{P(BC^c|AD^c)}_{p} \quad ,
\end{aligned}
$$

Given $AD^c \cap BC$, note that the event $A \leftrightarrow D$ is still true provided that at least one of the links AB and AC is present and at least one of the links BD and CD is present. In terms of the events and set operations, we require $(AB \cup AC) \cap (BD \cup CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC$. Therefore, again relying on independence,

$$
\begin{aligned}
P(A \leftrightarrow D|AD^c \cap BC) &= P((AB \cup AC) \cap (BD \cup CD)) = P(AB \cup AC)P(BD \cup CD) \\
&= (1 - P(AB^c \cap AC^c))(1 - P(BD^c \cap CD^c)) \\
&= (1 - \underbrace{P(AB^c)P(AC^c)}_{p^2})(1 - \underbrace{P(BD^c)P(CD^c)}_{p^2}) = \left(1 - p^2\right)^2 \quad .
\end{aligned}
$$

Given $AD^c \cap BC^c$, note that the event $A \leftrightarrow D$ is still true provided that both of the links AB and BD are present or both of the links AC and CD are present. In terms of the events and set operations, we require $(AB \cap BD) \cup (AC \cap CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC^c$. Thus,

$$
\begin{aligned}
P(A \leftrightarrow D|AD^c \cap BC^c) &= P((AB \cap BD) \cup (AC \cap CD)) \\
&= 1 - P((AB \cap BD)^c \cap (AC \cap CD)^c) \\
&= 1 - P((AB \cap BD)^c)P((AC \cap CD)^c) \\
&= 1 - (1 - P(AB \cap BD))(1 - P(AC \cap CD)) \\
&= 1 - (1 - \underbrace{P(AB)P(BD)}_{(1-p)^2})(1 - \underbrace{P(AC)P(CD)}_{(1-p)^2}) \\
&= 1 - \left(1 - (1-p)^2\right)^2 \quad .
\end{aligned}
$$

Finally, substituting these previous two answers into the above equations yields

$$
\boxed{P(A \leftrightarrow D) = 1 - p + \left\{\left(1 - p^2\right)^2 (1-p) + \left[1 - \left(1 - (1-p)^2\right)^2\right] p\right\} p}
$$

2. Using all the similar reasonings discussed in part (a),

$$
\begin{aligned}
P(A \leftrightarrow D|AB^c) &= \overbrace{P(A \leftrightarrow D|AB^c \cap AD)}^{1}\,\overbrace{P(AD|AB^c)}^{1-p} + \\
&\quad P(A \leftrightarrow D|AB^c \cap AD^c)\,\underbrace{P(AD^c|AB^c)}_{p}
\end{aligned}
$$

$$\begin{aligned}
P(A \leftrightarrow D | AB^c \cap AD^c) &= P(AC \cap (CD \cup (BC \cap BD))) \\
&= P(AC)\,(1 - P(CD^c \cap (BC \cap BD)^c)) \\
&= (1-p)\,(1 - P(CD^c)(1 - P(BC \cap BD))) \\
&= (1-p)\left(1 - p\left(1 - (1-p)^2\right)\right)
\end{aligned}$$

$$\boxed{P(A \leftrightarrow D | AB^c) = 1 - p + (1-p)\left\{1 - p\left[1 - (1-p)^2\right]\right\} p} \quad .$$

3. Again reasoning as in part (a), where we already computed $P(A \leftrightarrow D | AD^c \cap BC^c)$,

$$\begin{aligned}
P(A \leftrightarrow D | BC^c) &= \underbrace{P(A \leftrightarrow D | BC^c \cap AD)}_{1}\,\underbrace{P(AD)}_{1-p} + \underbrace{P(A \leftrightarrow D | AD^c \cap BC^c)}_{1-(1-(1-p)^2)^2}\,\underbrace{P(AD^c)}_{p} \\
&= \boxed{1 - p + \left\{1 - \left[1 - (1-p)^2\right]^2\right\} p} \quad .
\end{aligned}$$

4. All the required calculations were done in part (a):

$$\boxed{P(A \leftrightarrow D | AD^c) = \left(1 - p^2\right)^2 (1-p) + \left[1 - \left(1 - (1-p)^2\right)^2\right] p} \quad .$$

Problem 3.9

Consider an experiment where two distinct tetrahedral dice are rolled, each die has sides marked $\{1, 2, 3, 4\}$. The sample space is the sixteen element set:

$$\Omega = \{(1,1), (1,2), (1,3), (1,4), (2,1), \ldots, (4,4)\}$$

Consider the following events:

$$\begin{aligned}
A &= \text{The first die shows a } 1 = \{(1,1), (1,2), (1,3), (1,4)\} \\
B &= \text{The second die shows a } 1 = \{(1,1), (2,1), (3,1), (4,1)\}
\end{aligned}$$

Note that $A \cap B = \{(1,1)\}$. Let P be the discrete uniform probability law, i.e.

$$P((1,1)) = P((1,2)) = \cdots = P((4,4)) = \frac{1}{16}$$

and define another probability law Q to be

$$Q((x,y)) = \begin{cases} \frac{1}{64} & , \quad (x,y) = (1,1), (1,2), (2,1), (2,2) \\ \frac{5}{64} & , \quad \text{otherwise} \end{cases} \quad .$$

1. A and B are independent under P:

$$P(A)P(B) = \frac{4}{16}\frac{4}{16} = \frac{1}{16} = P(A \cap B).$$

2. A and B are dependent under Q:

$$Q(A)Q(B) = \frac{2*1+5*2}{64} * \frac{2*1+2*5}{64} = \frac{9}{256} \neq Q(A \cap B) = \frac{1}{64}.$$

3. Let C be the event that the first die x is equal to 1 or 2 *and* the second die y is equal to 1 or 2; in other words,

$$C = \{(1,1),(1,2),(2,1),(2,2)\} \quad .$$

Conditioning on C, note that

$$Q((1,1)|C) = Q((1,2)|C) = Q((2,1)|C) = Q((2,2)|C) = \frac{1}{4} \quad .$$

A and B are conditionally independent (given C) under Q:

$$Q(A|C)Q(B|C) = \left(2 \cdot \frac{1}{4}\right)\left(2 \cdot \frac{1}{4}\right) = \frac{1}{4} = Q(A \cap B|C) = \frac{1}{4} \quad .$$