EE126: Probability and Random Processes **SP'07**

Problem Set 4 — Due Feb, 15

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Problem 4.1. *Suppose three fair dice are rolled. What is the probability that at most one six appears?*

Solution:
We know that the probability that a six appears for one die is 1/6 and that event is independent to the outcome of the other dice.
Let X be the number of six that appear in this experiment. X can be modeled as a Binomial random variable with parameters $n = 3$ and $p = \frac{1}{6}$. We know that the Binomial distribution has pmf

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The probability that is asked in this question is

$$P(\text{at most one six}) = P(X \leq 1) = P(X = 0) + P(X = 1)$$

But we have

$$P(X = 0) = \binom{3}{0}(1/6)^0(5/6)^{3-0} = (5/6)^3 \quad \text{and} \quad P(X = 1) = 6\frac{1 \times 5^2}{6 \times 6^2}$$

Thus

$$P(\text{at most one six}) = \frac{151}{256}$$

Problem 4.2. *Suppose we are given a coin for which the probability of heads is* p $(0 < p < 1)$ *and the probability of tails is* $1 - p$. *Consider a sequence of independent flips of the coin.*

1. *Let* y $(y = 1, 2, ...)$ *be the number of flips up to and including the flip on which the first head occurs. Determine the pmf* $p_y(y_0)$ *for all values of* y_0.
2. *Let* x $(x = 0, 1)$ *be the number of heads that occur on any particular flip.*
   (a) *Determine E(x).*
   (b) *Determine* σ_x^2.
3. *Let* k $(k = 0, 1, ..., N)$ *be the number of heads that occur on the first* N *flips of the coin. Determine*
   (a) *the pmf* $p_k(k_0)$

(b) *$E(k)$ [Hint: Your results from part (b) may help you in determining $E(k)$ and σ_k^2.]*

(c) σ_k^2

4. *Given that a total of exactly six heads resulted from the first nine flips, what is the conditional probability that both the first and seventh flips were tails?*

5. *Let h be the number of heads that occur on the first twenty flips. Let C be the event that a total of exactly ten heads resulted from the first eighteen flips. Find*

   (a) $E[h|C]$

   (b) $\sigma_{h|C}^2$

Solution:
In this exercise all we need is to identify the distribution of random variables.

1. In this part we are looking for the time of the apparition of the first Head. This corresponds to a geometric random variable with parameter p. It has pmf

$$P[y = k] = (1-p)^{k-1}p \quad for \quad k = 1, 2, \ldots$$

2. X models a random variable that tells whether we obtain a success (Head) or a failure (Tail). It is known as a Bernoulli random variable with parameter p.
We know that the mean and variance are:

$$E[X] = 0.P(X = 0) + 1.P(X = 1) = 0 + p = p$$

$$\begin{aligned} Var(X) = \sigma_X^2 = E\left[(X-p)^2\right] &= (0-p)^2(1-p) + (1-p)^2 p \\ &= (1-p)p(p+1-p) = p(1-p) \end{aligned}$$

3. K is the number of Heads that occur on the first N flips. Again we know that this is modeled by a Binomial random variable with parameters (N, p).
The pmf is given by

$$P[K = n] = \binom{N}{n} p^n (1-p)^{N-n}$$

Writing K as the sum of independent Bernoulli random variables (whether we get a Head or not in each flip) we have

$$K = X_1 + X_2 + \cdots + X_N$$

Using the linearity of the expectation we have

$$E[K] = E[X_1 + X_2 + \cdots + X_N] = E[X_1] + E[X_2] + \cdots + E[X_N] = Np$$

Using the independence of the Bernoulli random variables, we obtain

$$Var(K) = Var(X_1 + X_2 + \cdots + X_N) = Var(X_1) + Var(X_2) + \cdots + Var(X_N) = Np(1-p)$$

4. We will use the definition of conditional probability.

$$\begin{aligned} P(X_1 = T, X_7 = T | K_9 = 6) &= \frac{P(X_1 = T, X_7 = T, K_9 = 6)}{P(K_9 = 6)} \\ &= \frac{P(K_9 = 6 | X_1 = T, X_7 = T) P(X_1 = T, X_7 = T)}{P(K_9 = 6)} \end{aligned}$$

First notice that X_1 and X_7 are independent, thus $P(X_1 = T, X_7 = T) = P(X_1 = T)P(X_7 = T) = (1-p)^2$.
Also given that the first and the seventh flips were Tails, the number of Heads is just the number of Heads that occur in 7 flips.

$$P(K_9 = 6 | X_1 = T, X_7 = T) = P(K_7 = 6) = \binom{7}{6} p^6 (1-p) = 7p^6(1-p)$$

Thus

$$P(X_1 = T, X_7 = T | K_9 = 6) = \frac{(1-p)^2 \times 7p^6(1-p)}{\binom{9}{6} p^6 (1-p)^3} = \frac{7}{3 \times 4 \times 7} = \frac{1}{12}$$

5. Let us compute $P(h = n|C)$

$$P(h = n|C) = \begin{cases} 0 & 0 \le n < 10 \\ p_n, & n \in \{10, 11, 12\} \\ 0 & n \ge 13 \end{cases}$$

where $p_n = P(\text{n-10 Heads in last 2 flips}) = \binom{2}{n-10} p^{n-10}(1-p)^{2-(n-10)}$
Now we can compute

$$E[h|n] = 10.(1-p)^2 + 11.2.p(1-p) + 12.p^2$$

To find the variance, we first compute

$$E[h^2|n] = 10^2.(1-p)^2 + 11^2.2.p(1-p) + 12^2.p^2$$

Thus the variance is

$$\sigma^2_{h|C} = 100.(1-p)^2 + 121.2.p(1-p) + 144.p^2 - (10.(1-p)^2 + 11.2.p(1-p) + 12.p^2)^2$$

Problem 4.3. *Let X be a random variable with density function*

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, x < 0 \end{cases}$$

1. *Find c.*

2. *What is $P[X > 2]$?*
3. *Compute $E[X]$.*

Solution:

1. To find c, we use the fact that the density of a random variable integrates to 1.

$$\int_0^\infty ce^{-2x}dx = 1 = c\left[-\frac{1}{2}e^{-2x}\right]_0^\infty = \frac{c}{2}$$

which implies that $c = 2$

2.

$$P[X > 2] = \int_2^\infty 2e^{-2x}dx = e^{-4}$$

3.

$$E[X] = \int_0^\infty 2xe^{-2x}dx$$

Using integration by parts with $u = x$ and $v' = e^{-2x}$ we get

$$E[X] = \left[-2xe^{-2x}\right]_0^\infty + \int_0^\infty e^{-2x}dx = \frac{1}{2}$$

Problem 4.4. *A marksman takes 10 shots at a target and has probability 0.2 of hitting the target with each shot. Let X be the number of hits.*

1. *Calculate and plot the PMF of X.*
2. *What is the probability of scoring no hits?*
3. *What is the probability of scoring more hits than misses?*

Solution:

1. It is easy to recognize that X is Binomial$(10, 0.2)$. Thus the pmf is

$$P[X = k] = \binom{10}{k}(.2)^k(.8)^{10-k}$$

Figure 4.1 shows a plot of the pmf.

2. The probability of scoring no hits is $P(X = 0) = (.8)^{10} = 0.1074$

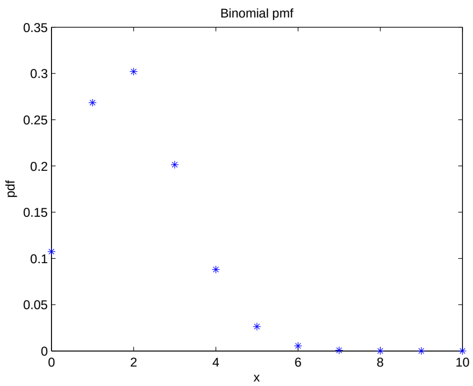

Figure 4.1. Binomial(10,.2) pmf

3. The probability of scoring more hits than misses is

$$P(X > 5) = \sum_{k=6}^{10} \binom{10}{k} (.2)^k (.8)^{10-k} = 0.0064$$

Problem 4.5. *Your probability class has 250 undergraduate students and 50 graduate students. The probability of an undergraduate (or graduate) student getting an A is 1/3 (or 1/2, respectively). Let X be the number of students that get an A in your class.*

1. *Find the PMF of X.*
2. *Calculate $E[X]$ using the total expectation theorem, rather than the PMF of X.*
3. *Calculate $E[X]$ and var(X) by viewing X as a sum of random variables, whose statistics are easily calculated.*

Solution:
Define the following two random variables:
Y = number of undergraduate students who gets an A
Z = number of graduate students who gets an A

The PMF for the random variables Y and Z are:

$$\begin{aligned} p_Y(y) &= \binom{250}{y} (\frac{1}{3})^y (\frac{2}{3})^{250-y} \\ p_Z(z) &= \binom{50}{z} (\frac{1}{2})^{50} \end{aligned}$$

1. We wish to find the PMF of X define $X = Y + Z$

$$\begin{aligned} p_X(x) &= \sum_{(y,z)|(y+z)=x} p_{Y,Z}(y,z) \\ &= \sum_{(y,z)|(y+z)=x} p_Y(y)p_Z(z) \quad (Y \text{ and } Z \text{ are independent}) \\ &= \begin{cases} \sum_{i=0}^{min(x,50)} \binom{50}{i}(\frac{1}{2})^{50}\binom{250}{x-i}(\frac{1}{3})^{x-i}(\frac{2}{3})^{250-x+i} & 0 \leq x \leq 250 \\ \sum_{i=x-250}^{50} \binom{50}{i}(\frac{1}{2})^{50}\binom{250}{x-i}(\frac{1}{3})^{x-i}(\frac{2}{3})^{250-x+i} & 251 \leq x \leq 300 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Notice how closely the pmf of X defined as a function of the pmf of Y and Z resembles the convolution operation.

2. We wish to find $E[X]$ using the total expectation theorem. So we partition the class into the disjoint events Undergraduates and Graduates. Let $X = X_1 + X_2 + \cdots + X_{300}$, where X_i are defined as follows:

$$\begin{aligned} X_i &= \begin{cases} 1 & \text{if student i gets A} \\ 0 & \text{if student i does not get A} \end{cases} \\ E[X] &= E[\sum_{i=1}^{300} X_i] \end{aligned}$$

The X_i are independent, with the following conditional PMF.

$$\begin{aligned} P(X_i|Undergrad) &= \begin{cases} \frac{1}{3} & X_i = 1 \\ \frac{2}{3} & X_i = 0 \\ 0 & \text{otherwise} \end{cases} \\ P(X_i|Grad) &= \begin{cases} \frac{1}{2} & X_i = 1 \\ \frac{1}{2} & X_i = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} E[X] &= 300E[X_i] \\ &= 300(E[X_i|Undergrad]P(Undergrad) + E[X_i|Grad]P(Grad)) \\ &= 300((\frac{1}{3})(\frac{5}{6}) + (\frac{1}{2})(\frac{1}{6})) \\ &= \frac{325}{3} \end{aligned}$$

3. Here we find the $E[X]$ and $var(X)$ by viewing X as a sum of the random variables Y and Z as defined above.

$$\begin{aligned} X &= Y + Z \\ E[X] &= E[Y] + E[Z] \quad \text{(by linearity of expectation)} \\ &= 250(\frac{1}{3}) + 50(\frac{1}{2}) \\ &= \frac{325}{3} \end{aligned}$$

$$\begin{aligned} var(X) &= var(Y) + var(Z) \quad \text{(from independence of } Y \text{ and } Z) \\ &= 250(\frac{2}{9}) + 50(\frac{1}{4}) \\ &= \frac{1225}{18} \end{aligned}$$

Problem 4.6. *Consider the random variable X with PMF*

$$p_X(x) = \begin{cases} \frac{x^2}{a} & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

1. *Find a and $\mathbf{E}[X]$.*
2. *What is the PMF of the random variable $Z = (X - \mathbf{E}[X])^2$?*
3. *Using part (b) compute the variance of X.*

Solution:

1. The sum of the PMF of a random variable over the possible values that it can take must be equal to 1. Hence, we have

$$1 = \sum_{x=-3}^{3} p_X(x) = \frac{9}{a} + \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} = \frac{28}{a},$$

which implies that $a = 28$. The expected value of random variable is given by

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=-3}^{3} x \cdot \frac{x^2}{a} = -3 \cdot \frac{9}{a} - 2 \cdot \frac{4}{a} - 1 \cdot \frac{1}{a} + 1 \cdot \frac{1}{a} + 2 \cdot \frac{4}{a} + 3 \cdot \frac{9}{a} = 0.$$

(In fact, the expected value of a random variable is always equal to 0 if its PMF is even [i.e. $p_X(x) = p_X(-x)$ for all x], and it is a simple exercise to confirm this.)

2. The following table shows the value of Z for a given value of X and the probability of that event.

x	-3	-2	-1	0	1	2	3
$p_X(x)$	9/28	1/7	1/28	0	1/28	1/7	9/28
$Z\|X = x$	9	4	1	0	1	4	9

We see that Z can take only three possible values with non-zero probability, namely 1,4, and 9. In addition, for each value, there corresponds two values of X. So we have, for example, $p_Z(9) = \mathbf{P}(Z = 9) = \mathbf{P}(X = -3) + \mathbf{P}(X = 3) = p_X(-3) + p_X(3)$. Hence the PMF of Z is given by

$$p_Z(z) = \begin{cases} 1/14 & \text{if } z = 1, \\ 2/7 & \text{if } z = 4, \\ 9/14 & \text{if } z = 9, \\ 0 & \text{otherwise.} \end{cases}$$

3. Recall that $\text{var}(X) = \mathbf{E}[(X - \mathbf{E}[X])]^2$, so $\text{var}(X) = \mathbf{E}[Z] = 1 \cdot \frac{1}{14} + 4 \cdot \frac{2}{7} + 9 \cdot \frac{9}{14} = 7$.

Problem 4.7. *Prove that*

$$E[X^2] \geq (E[X])^2 .$$

When do we have equality?

Solution:
Now that we have seen the definition of the variance and that we know that $X \geq 0 \Rightarrow E[X] \geq 0$ we can write

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2 \geq 0$$

then

$$E[X^2] \geq (E[X])^2$$

We have equality when $Var(X) = 0$, that is, when X is constant.

Problem 4.8. *A coin having probability p of coming up heads is successively flipped until the rth head appears. Argue that X, the number if flips required, will be n, $n \geq r$, with probability*

$$P[X = n] = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

Compute the expectation of X.

Solution:
We know that if $X = k$, then there are $r-1$ Heads in the first $k-1$ flips and the k'th flip comes up Head. Note that $k \geq r$ (we need to flip at least r times!)
Now look at the first $k-1$ flips as an experiment we we flip $k-1$ times the coin and record the number of Heads. As we know now, this experiment corresponds to a Binomial random variable with parameters $(k-1, p)$. Thus the probability of having $r-1$ Heads is given by

$$P(r-1 \text{ Heads out of } k-1 \text{ flips}) = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-1-(r-1)}$$

Now using the fact that the k'th flip comes up Head and is independent to the previous flips, we obtain

$$P(X = k) = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} p = \binom{k-1}{r-1} p^{r}(1-p)^{k-r}$$

This random variable is called *negative binomial.*
The simplest way to compute the expectation of the negative binomial random variable is to write it as the sum of geometric random variables. In fact, since each coin flip is independent to the others, the time of the r'th Head can be written as

$$X = \sum_{i=1}^{r} T_i$$

where the T_i's are independent and identical random variables each modeling the time of the first Head (see X as X=time for first arrival+time for the next first arrival+time for next arrival....)
Now we know the mean the time of the first arrival (geometric random variable)...$E[T_i] = \frac{1}{p}, i = 1, 2, \ldots$.
Using the linearity of the expectation we obtain

$$E[X] = E\left[\sum_{i=1}^{r} T_i\right] = \sum_{i=1}^{r} E[T_i] = \sum_{i=1}^{r}(1/p) = \frac{r}{p}$$