UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 5
Spring 2007

Issued: Thursday, March 8, 2007 **Due:** Thursday, March 15, 2007

Problem 5.1

1. $N = 200,000$.
2. $N = 100,000$.

Problem 5.2

a)This is a straight application of the Chebychev inequality. Chebychev tells us that: $P(|X-7| \geq 3) \leq \frac{Var}{3^2} = \frac{2}{3}$ and therefore that $P(4 < X < 10) \geq \frac{1}{3}$.

b)If the variance $= 9$, $P(|X-7| \geq 3) \leq \frac{Var}{3^2} = 1$ and therefore that $P(4 < X < 10) \geq 0$.

Problem 5.3

(a)

$$p_{N,K}(n,k) = \begin{cases} 1/4k & \text{if } k = 1,2,3,4 \text{ and } n = 1,\ldots,k \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48 & n = 1 \\ 1/8 + 1/12 + 1/16 = 13/48 & n = 2 \\ 1/12 + 1/16 = 7/48 & n = 3 \\ 1/16 = 3/48 & n = 4 \\ 0 & \text{otherwise} \end{cases}$$

(c) The conditional PMF

$$p_{K|N}(k|2) = \frac{p_{N,K}(2,k)}{p_N(2)} = \begin{cases} 6/13 & k = 2 \\ 4/13 & k = 3 \\ 3/13 & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

(d) Let A be the event that Chuck bought at least 2 but no more than 3 books, $E[K|A] = 3$ $var(K|A) = \frac{3}{5}$

(e) $E[T] = \frac{21}{4}$

Problem 5.4

1.
$$p_X(x) = \begin{cases} \frac{1}{20} & , \quad x = 1 \\ \frac{3}{20} & , \quad x = 2 \\ \frac{3}{10} & , \quad x = 3 \\ \frac{1}{2} & , \quad x = 4 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad , \qquad \mathbf{E}[X] = \frac{13}{4}$$

2.
$$p_W(w) = \begin{cases} \frac{1}{10} & , \quad w = 2 \\ \frac{1}{20} & , \quad w = 3 \\ \frac{7}{20} & , \quad w = 4 \\ \frac{3}{10} & , \quad w = 6 \\ \frac{1}{5} & , \quad w = 8 \\ 0 & , \quad \text{otherwise} \end{cases}$$

3. $\mathbf{E}[R] = \frac{3}{4}$, $\text{var}(R) = \frac{63}{80}$

4. $\sigma_{R|A} = \frac{\sqrt{3}}{4}$

Problem 5.5

1. $\mu = 3$, $\sigma^2 = 2$

2. $\mu = 5$, $\sigma^2 = 20$.

Problem 5.6

1. To calculate the probability of heads on the next flip, we use the continuous version of the total probability theorem:

$$\begin{aligned} P(H) &= \int_0^1 P(H|P = p) \cdot f_P(p) dp \\ &= \int_0^1 p^2 e^p dp \\ &= e - 2 \end{aligned}$$

2. Here we need to use the continuous version of Bayes' theorem:

$$\begin{aligned} f_{P|A}(p|A) &= \frac{P(A|p)f_P(p)}{\int_0^1 P(A|p)f_P(p)dp} \\ &= \begin{cases} \frac{p^2 e^p}{e-2}, & 0 \le p \le 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

3. Now we apply the above result to the technique used in (a):

$$\begin{aligned} P(H) &= \int_0^1 P(A|P=p) \cdot f_{P|A}(p|A)dp \\ &= \frac{1}{e-2}\int_0^1 p^3 e^p dp \\ &= \frac{1}{e-2} \cdot (6-2e) \\ &= \frac{.564}{.718} \\ &\approx .786 \end{aligned}$$

Problem 5.7

Note that we can rewrite $E[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots]$ as follows:

$$\begin{aligned} & E[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots] \\ = \; & E[X_1 \mid S_n = s_n, X_{n+1} = s_{n+1} - s_n, X_{n+2} = s_{n+2} - s_{n+1}, \ldots] \\ = \; & E[X_1 \mid S_n = s_n], \end{aligned}$$

where the last equality holds due to the fact that the X_i's are independent.

We also note that

$$E[X_1 + \cdots + X_n \mid S_n = s_n] = E[S_n \mid S_n = s_n] = s_n$$

It follows from the linearity of expectations that

$$E[X_1 + \cdots + X_n \mid S_n = s_n] = E[X_1 \mid S_n = s_n] + \cdots + E[X_n \mid S_n = s_n]$$

Because the X_i's are identically distributed, we have the following relationship:

$$E[X_i \mid S_n = s_n] = E[X_j \mid S_n = s_n], \text{ for any } 1 \le i \le n, 1 \le j \le n.$$

Therefore,

$$E[X_1 + \cdots + X_n \mid S_n = s_n] = nE[X_1 \mid S_n = s_n] = s_n \quad \Rightarrow \quad E[X_1 \mid S_n = s_n] = \frac{s_n}{n}.$$

Problem 5.8

a) γ is determined because the density function must integrate to 1. Since (X, Y) uniformly distributed in R, we have:

$$\int\int \gamma dxdy = 1$$

where the integral is over the area of R. Therefore $\frac{1}{\gamma}$ =Area R.
b) Showing independence is equivalent to showing that:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

But this is clear, since:

$$f_{X,Y}(x, y) = f_Y(x) = f_Y(y) = 1$$

c) Consider a region that consists of the upper half plane, and the point $(1, -1)$. If we are told that $Y < 0$, X is determined, and hence X, Y cannot be independent in this region.
d) To find the probability that (X, Y) lie in the circle C inscribed in the R in part (b) we could integrate, or observe that the integral will in fact come out to the area of the circle, and hence the desired probability will be the ratio of the are of the circle to the area of the square:

$$P((X, Y) \in C) = \frac{.25\pi}{1}.$$