EEL 5544 Lecture 19

GENERATING RANDOM VARIABLES

- To generate a random variable with an arbitrary distribution, we would like to:
  1. Generate a Uniform random variable on $(0, 1]$, U
  2. Apply a function g to U such that if $X = g(U)$, then X has the desired distribution

- We begin by making an observation: Suppose X is a random variable with distribution function $F_X(x)$

  Then what is the distribution of $Y = F_X(X)$?

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

and

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1, & y \geq 1 \end{cases}$$

By inspection Y is a ______________________ random variable!

- Thus to generate a random variable X with distribution function $F_X(x)$, we can use the following procedure:

- **Transformation Method**

  To generate a RV X with a **continuous distribution**:

  1. Generate a random variable U that is distributed uniform on $[0, 1]$ using commonly available methods.
  2. Let $X = F_X^{-1}(U)$

Proof:

It is a notational nightmare if we straight away let $X = F_X^{-1}(U)$ so instead, let's first just let $Z = F_X^{-1}(U)$

Then

$$
\begin{aligned}
F_Z(z) &= P\left(F_X^{-1}(U) \leq z\right) \\
&= \\
&= \\
&=
\end{aligned}
$$

because

So Z has the desired distribution. Replacing Z with X finishes the proof.

Example: Generate a random variable X that has an exponential distribution with parameter λ

To generate a RV X with a **discrete distribution** on a consecutive subset of the integers:

1. Generate a random variable U that is distributed uniform on $[0, 1]$ using commonly available methods.

2. Let $X = k$ if $F_X(k-1) < U \leq F_X(k)$.

Proof:

Again, in order to avoid confusing notation, let's let $Z = k$ if $F_X(k-1) < U \leq F_X(k)$.

$$P(Z = k) \quad = \quad$$

$$=$$

which is the desired probability mass at point k

Again, replace Z with X, and the proof is complete.

FUNCTIONS OF MULTIPLE RANDOM VARIABLES:

ONE FUNCTION OF SEVERAL RANDOM VARIABLES

- We often have situations in which we are interested in a function that involves two or more random variables
- For instance, if X and Y are random variables, then we may be interested in the following:
  - The signal X is received in the presence of additive noise Y, $Z = X + Y$
  - A device has two identical components. Let X and Y be the time until each component fails. Let Z be the time until the device stops working, which can be:
    - Only when both components fail: $Z = \max(X, y)$
    - When either component fails: $Z = \min(X, Y)$
  - A random signal is modulated by another signal, $Z = XY$.
  - The Euclidean distance of a point in a plane is $Z = \sqrt{X^2 + Y^2}$
- I'll use a more general notation than the book's notation at this point.

  Let the random variables that are input to the function g be denoted by

  $$X_1, X_2, \ldots, X_n = \mathbf{X_n}$$

  Then $Z = g(\mathbf{X_n})$

- The solutions to problems of the from $Z = g(\mathbf{X_n})$ are not fundamentally different from the solutions to problems of the form $Z = g(X)$.

  We just have to be a little more careful.

  Consider the distribution function for Z,

$$F_Z(z) = P\left[g(\mathbf{X_n}) \le z\right]$$

  Let $R_z = \{\mathbf{x_n} | g(\mathbf{x_n}) \le z\}$. Then

$$F_Z(z) = P\left[\mathbf{X_n} \in R_z\right]$$

  The problem is that the region R_z is not necessarily rectangular, in which case the probability of $\mathbf{X_n} \in R_z$ cannot be directly calculated from the distribution function

  However, the probability of any region can be calculated by integrating the density over that region:

$$F_Z(z) = \int \cdots \int_{\mathbf{x_n} \in R_z} f_{\mathbf{X_n}}(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

  This is best illustrated by an example:

  Example: Let $Z = X + Y$. Find the distribution and density functions for Z in terms of the joint density function for X and Y.

- Note that if X and Y are s.i., then

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= [f_X * f_Y](z), \end{aligned}$$

where $*$ represents the ____________________ operator

Example: X and Y are independent exponential random variables, each with parameter $\lambda = 1$

- Applying this technique repeatedly for sums of multiple random variables would be difficult. We will later investigate more powerful techniques to deal with sums of multiple random variables.

USING CONDITIONAL PDFS TO FIND THE PDF OF A FUNCTION OF SEVERAL RVS

- Let $Z = g(X, Y)$
- If we condition on $Y = y$, then $g(X, y)$ is a function of only one RV, so we can use the techniques from the previous sections to find

$$f_{Z|Y}(z|Y = y)$$

- Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_Y(y) dy$$

by the Law of Total Probability