EXPECTED VALUE OF FUNCTIONS OF RVS

- If $Z = g(X,Y)$, then

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dx\,dy.$$

EX: Expected value of Sum of RVs
Let $Z = X + Y$, where X and Y not necessarily s.i. Then

$$\begin{aligned}
E[Z] &= E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\,dx\,dy \\
&= \\
&=
\end{aligned}$$

- In general,

$$E\left[\sum_{i=1}^{N} X_i\right] =$$

- If $g_i(\underline{X}) = g_i(X_i)$ for all i, then

$$E\left[\sum_{i=1}^{N} a_i g_i(\underline{X})\right]$$

- If X, Y are s.i. and $\underline{g}(X,Y) = g_1(X)g_2(Y)$, then

$$\begin{aligned}
E\left[g(X,Y)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)\,g_2(y)\,f_X(x)\,f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} g_1(x)\,f_X(x)\,dx \int_{-\infty}^{\infty} g_2(y)\,f_Y(y)\,dy \\
&=
\end{aligned}$$

- In general, if X_i are mutually s.i., then

$$E\left[\prod_{i=1}^{N} g_i(X_i)\right] = \prod_{i=1}^{N} E\left[g_i(X_i)\right] \tag{1}$$

- Note, however, that (1) being true does not imply that X_i are mutually s.i. (it is not a sufficient condition)

CONDITIONAL EXPECTED VALUE

DEFN The *conditional expected value of* Y *given* $X = x$ is

$$E[Y|X=x] = E[Y|x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy.$$

- Note that for each value of x, $E[Y|x]$ can be a different value
- So $g(x) = E[Y|x]$ is a ______________ of x.

⇒ If X is a random variable, then $g(X) = E[Y|X]$ is a ____________________________.

- Note that if $E[Y|X]$ is a ____________________________, then we can find the expected value of it:

$$E\Big[E\left[Y|X\right]\Big] = \int_{-\infty}^{\infty} E[Y|x] f_X(x)dx.$$

(note that we have $E[Y|x]$ in the above integral).

$$\begin{aligned} E\Big[E\left[Y|X\right]\Big] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy f_X(x)dx \\ &= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)dxdy \\ &= \\ &= \end{aligned}$$

- This also holds for functions of Y:

$$E\Big[E\left[h(Y)|X\right]\Big] = E[h(Y)]$$

For example,

$$E\Big[E\left[Y^k|X\right]\Big] = E[Y^k]$$

Ex 4.26 Find the mean and variance of the number of customer arrivals N during service time T of a specific customer, where:

  - T is an exponential RV with parameter α
  - The number of arrivals during time t is Poisson RV with parameter βt

By conditioning on $T = t$, the expected number of arrivals is

$$\begin{aligned} & E[N|T=t] = \beta t \\ \text{and} \qquad & E[N^2|T=t] = \beta t + (\beta t)^2. \end{aligned}$$

Then $E[N|T]$ is a random variable, and the mean of N is given by

$$\begin{aligned} E[N] &= E\Big[E\left[N|T\right]\Big] \\ &= \int_{-\infty}^{\infty} E[N|T=t] f_T(t) dt \\ &= \int_{0}^{\infty} (\beta t)\left[\alpha e^{-\alpha t}\right] dt \\ &= \beta E[T] = \frac{\beta}{\alpha} \end{aligned}$$

Similarly,

$$\begin{aligned} E[N^2] &= \int_{0}^{\infty} E[N^2|T=t] f_T(t) dt \\ &= \beta E[T] + \beta^2 E[T^2] \\ &= \frac{\beta}{\alpha} + \beta^2 \left(\frac{2}{\alpha}\right) \end{aligned}$$

Thus,

$$\text{Var}\left\{N\right\} = \frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha}$$

JOINT MOMENTS

DEFN The j,k*th joint moment* of X and Y is

$$E[X^jY^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x,y)\,dx\,dy.$$

DEFN The j,k*th central moment* of X and Y is

$$E[(X-\mu_X)^j(Y-\mu_Y)^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)^j(y-\mu_Y)^k f_{XY}(x,y)dx\,dy.$$

DEFN The *covariance* of X and Y is the $(1,1)$th central moment

$$\text{Cov}[X,Y] = \qquad (3)$$

- The covariance is usually calculated by simplifying equation (3):

$$\begin{aligned}\text{Cov}[X,Y] &= \\ &= \\ &= \end{aligned}$$

- Note that if either $\mu_X = 0$ or $\mu_Y = 0$, then $\text{Cov}[X,Y] =$ ____________.
- If X,Y are s.i., then

$$\begin{aligned}\text{Cov}[X,Y] &= \\ &= \\ &= \end{aligned}$$

DEFN The *correlation coefficient* of X and Y is

$$\rho_{XY} \quad =$$

- Note that $-1 \leq \rho_{XY} \leq 1$

Proof: Note that

$$E\left\{\left[\frac{X-\mu_X}{\sigma_X} \pm \frac{Y-\mu_Y}{\sigma_Y}\right]^2\right\} \geq 0$$

(because anything squared is ≥ 0*).*

Expanding this expression yields

$$\begin{aligned} E\left[\frac{(X-\mu_X)^2}{\sigma_X^2}\right] \quad \pm \quad 2E\left[\frac{(X-\mu_X)(Y-\mu_Y)}{\sigma_X\sigma_Y}\right] \\ +E\left[\frac{(Y-\mu_Y)^2}{\sigma_Y^2}\right] \geq 0 \end{aligned}$$

Thus

$$\begin{aligned} & 2\left(1 \pm \rho_{XY}\right) \geq 0 \\ \Rightarrow \quad & |\rho_{XY}| \leq 1 \end{aligned}$$

- ρ_{XY} is a measure of the dependence between X and Y
  - If X,Y are linearly related by $Y = aX + b$, then

$$\begin{aligned} \text{Cov}(X,Y) &= E[XY] - \mu_X\mu_Y \\ &= E[aX^2 + bX] - \mu_X(a\mu_X + b) \\ &= aE[X^2] + b\mu_X - a\mu_X^2 - b\mu_X \\ &= a\left(E[X^2] - \mu_X^2\right) \\ &= a\sigma_X^2. \end{aligned}$$

Thus the correlation coefficient is

$$\begin{aligned}\rho_{XY} &= \frac{a\sigma_X^2}{\sigma_X\sigma_Y} = \frac{a\sigma_X^2}{\sigma_X|a|\sigma_Y} \\ &= \text{sgn}(a)\end{aligned}$$

- If X,Y are s.i. , then $\rho_{XY} = 0$

DEFN RVs X and Y are *uncorrelated* if ________________.

- If X,Y are _____, then X,Y are uncorrelated. Note that the converse is ______________.

DEFN RVs X and Y are ____________________ if $E[XY] = 0$.

- Note that if X and Y are both ____________________ and ______________________ then at least one of $\mu_X = 0$, $\mu_Y = 0$

Example on board.