EEL 5544 Noise in Linear Systems Lecture 24

General Bivariate Gaussian Distribution

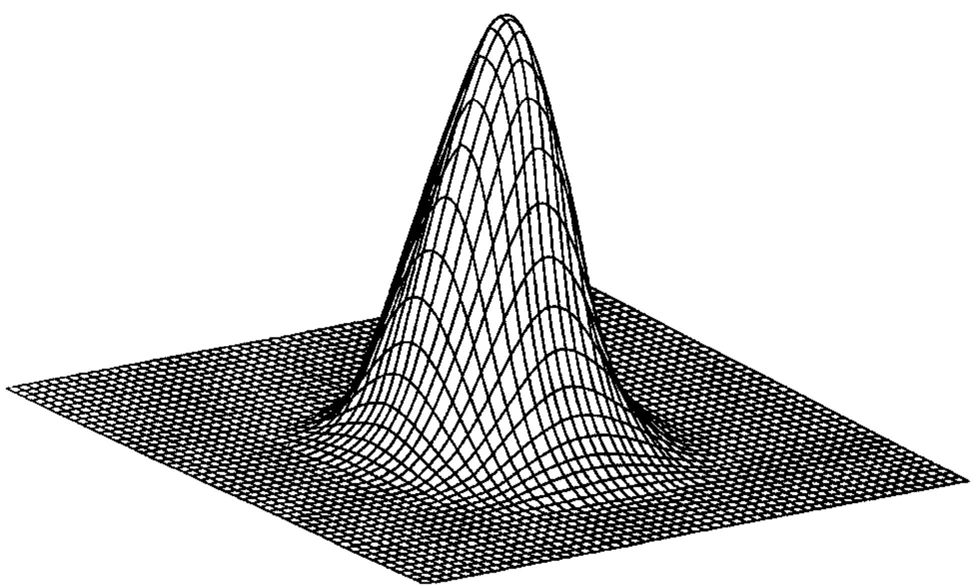
X, Y are *jointly Gaussian* if and only if the joint density of X and Y can be written as

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \exp\left\{ \frac{-1}{2(1-\rho_{X,Y}^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 -2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right] \right\}$$

- An equivalent condition that may be easier to work with is:

X and Y are jointly Gaussian if and only if $aX + bY$ is a Gaussian random variable for any real a and b

- pdf is centered at (μ_X, μ_Y)
- pdf is bell-shaped:



- Additional insight can be gained from considering contours of equal prob. density

  For equal prob.:

$$\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] = \text{const.} \quad (1)$$

- Equation (1) is the equation for an ellipse:

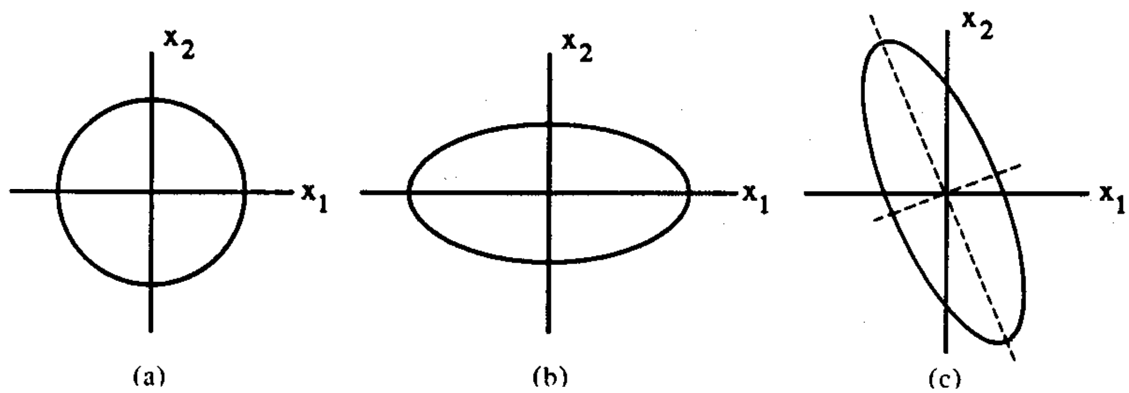
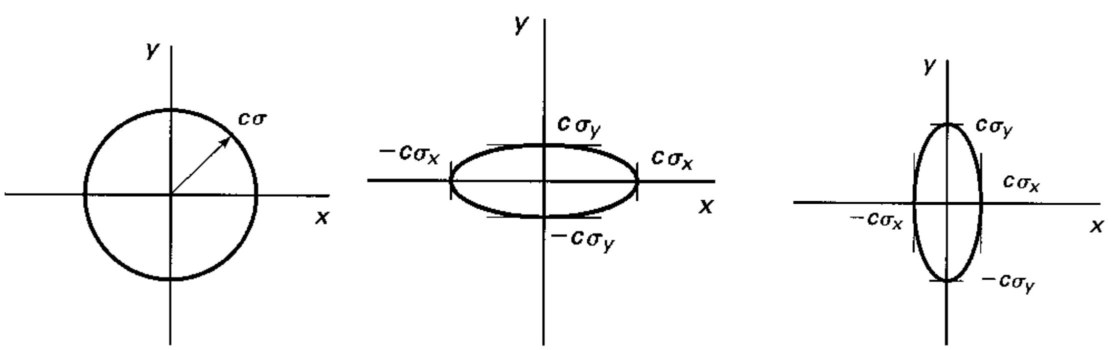


Fig. 4.4.1. Equal-probability contours for two Gaussian random variables: (a) uncorrelated equal variance; (b) uncorrelated unequal variance; (c) correlated unequal variance.

(From Komo, *Random Signal Analysis...*)

-When $\rho_{X,Y} = 0$, X and Y are s.i., and equal-prob. contour ellipse is aligned w/ x- and y-axes:



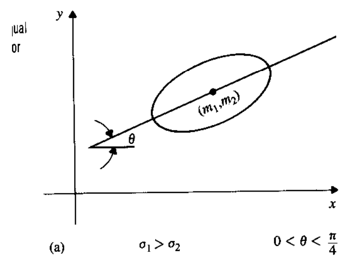
(a) $\sigma_x = \sigma_y; \rho_{XY} = 0$ (b) $\sigma_x > \sigma_y; \rho_{XY} = 0$ (c) $\sigma_x < \sigma_y; \rho_{XY} = 0$

(From Stark and Woods, *Probability and Random Processes...*)

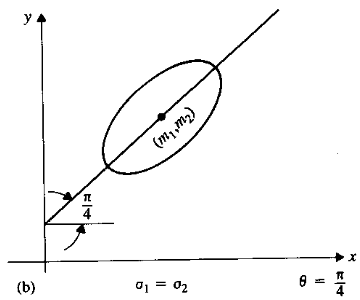
-When $\rho_{X,Y} \neq 0$,the major axis is at an angle given by

$$\theta = \frac{1}{2} \arctan \left(\frac{2\rho_{X,Y}\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2} \right)$$

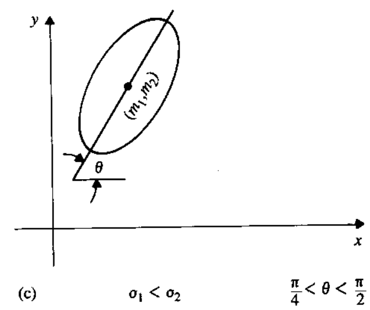
Note that $\sigma_X = \sigma_Y \Rightarrow \theta = 45$ degrees



(d)

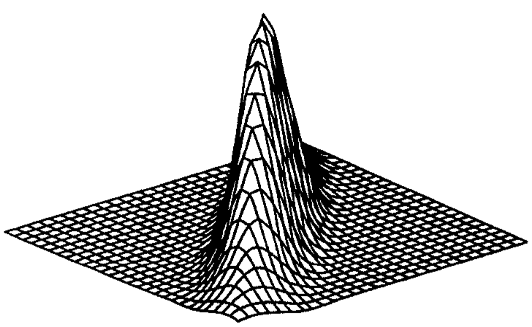


(e)

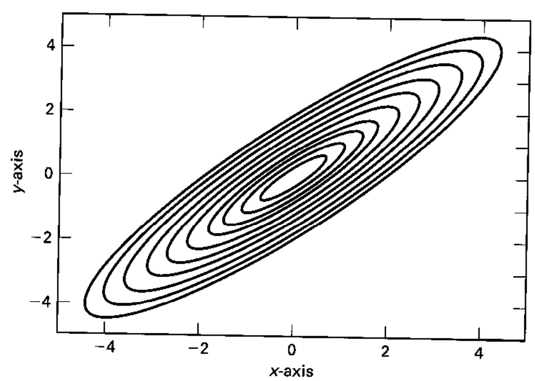


(f)

Joint Gaussian RVs, $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 2$, $\rho_{XY} = 0.9$



(g) pdf



(h) equiprobability contours

(From Stark and Woods, *Probability and Random Processes...*)

SPECIAL CASE: JOINTLY GAUSSIAN RANDOM VARIABLES WITH ZERO MEAN AND UNIT VARIANCE

DEFN Two Gaussian random variables X and Y that each have mean 0 and variance 1 are said to be *jointly Gaussian* if their joint density function can be written as

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\quad \times \exp\left\{\frac{-\left(x^2-2\rho xy+y^2\right)}{2\left(1-\rho^2\right)}\right\}, \\ &\quad -\infty < x < \infty \\ &\quad -\infty < y < \infty \end{aligned}$$

- The marginal pdfs can be found by completing the square.

Ex: Find the marginal pdf of X:

The marginal density for X is given by

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-\left(x^2-2\rho xy+y^2\right)}{2\left(1-\rho^2\right)}\right\} dy$$

Take the argument of the exponential and complete the square in y:

$$\begin{aligned} x^2-2\rho xy+y^2 &= y^2-2\rho xy+\rho^2x^2+x^2-\rho^2x^2 \\ &= (y-\rho x)^2+x^2-\rho^2x^2 \end{aligned}$$

Then the marginal density for X is

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2(1-\rho^2)}{2(1-\rho^2)}\right\} \\ &\quad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{\frac{-\left(y-\rho x\right)^2}{2\left(1-\rho^2\right)}\right\} dy \end{aligned}$$

Letting $\sigma^2 = 1-\rho^2$, $\mu = \rho x$,

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\} \\ &\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-\left(y-\mu\right)^2}{2\sigma^2}\right\} dy \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\} \end{aligned}$$

- Therefore, we have shown that X is Gaussian with $\mu_X = 0$ and $\sigma_X^2 = 1$
- Similarly, Y is Gaussian with $\mu_Y = 0$ and $\sigma_Y^2 = 1$
- Note that X and Y can each be Gaussian without being jointly Gaussian.

  Ex: If the joint density of X and Y is given by

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi} \exp\left\{ \frac{-(x^2+y^2)}{2} \right\} \\ &\quad \times \left(1 + xy \exp\left\{-(x^2+y^2-2)\right\}\right), \end{aligned}$$

  then X and Y are each Gaussian but clearly not jointly Gaussian.

- Under what conditions are mean 0, variance 1, jointly Gaussian RVs statistically independent?

  We know that if two RVs X and Y are s.i., then

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) f_Y(y) \\ \Rightarrow \frac{1}{2\pi\sqrt{1-\rho^2}} &\exp\left\{ \frac{-\left(x^2 - 2\rho xy + y^2\right)}{2\left(1-\rho^2\right)} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-x^2}{2} \right\} \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-y^2}{2} \right\}, \end{aligned}$$

  which occurs if and only if $\rho = 0$.

- In other words, *uncorrelated* jointly Gaussian random variables are also *statistically independent*