EEL 5544 Noise in Linear Systems Lecture 2a

SET OPERATIONS

- Want to determine probabilities of events (combinations of outcomes)
- Each event is a set $\Rightarrow$ need operations on sets:

1. $A \subset B$ (read "A is a subset of B"):

DEFN The *subset operator* $\subset$ is defined for two sets A and B by

$$A \subset B \text{ if } x \in A \Rightarrow x \in B$$

Note that $A = B$ is included in $A \subset B$

$A = B$ if and only if $A \subset B$ and $B \subset A$
(useful for proofs)

2. $A \cup B$ (read "A union B" or "A or B")

DEFN The *union* of A and B is a set defined by

$$A \cup B = \{x \,|x \in A \text{ or } x \in B\}$$

3. $A \cap B$ (read "A intersect B" or "A and B")

DEFN The *intersection* of A and B is defined by

$$A \cap B = AB = \{x \,|x \in A \text{ and } x \in B\}$$

4. A^c or $\overline{A}$ (read "A complement")

DEFN The *complement* of a set A in a sample space Ω is defined by

$$A^c = \{x \,|x \in \Omega \text{ and } x \notin A\}$$

- Use Venn diagrams to convince yourself of:
  1. $(A \cap B) \subset A$ and $(A \cap B) \subset B$
  2. $\overline{(\overline{A})} = A$
  3. $A \subset B \Rightarrow \overline{B} \subset \overline{A}$
  4. $A \cup \overline{A} = \Omega$
  5. $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (DeMorgan's Law 1)
  6. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (DeMorgan's Law 2)

- One of the most important relations for sets is when sets are **mutually exclusive**

DEFN Two sets A and B are said to be *mutually exclusive* or *disjoint* if ____________________.

- **This is very important, so I will emphasize it again: mutually exclusive is a ____________________.**

SAMPLE SPACES

- Given the following experiment descriptions, write a mathematical expression for the sample space. Compare and contrast the experiments and sample spaces.
  1. Roll a fair 6-sided die and note the number on the top face.
  2. Roll a fair 6-sided die and determine whether the number on the top face is even or odd.

3. Toss a coin 3 times and note the sequence of outcomes.

4. Toss a coin 3 times and note the number of heads.

5. Roll a fair die 3 times and note the number of even results.

6. Roll a fair 6-sided die 3 times and note the number of sixes.

7. Simultaneously toss a quarter and a dime and note the outcomes?

8. Simultaneously toss two identical quarters and note the outcomes?

9. Pick a number at random between zero and one, inclusive.

10. Measure the lifetime of a computer chip.

MORE ON EVENTS

- Recall that an event A is a subset of the sample space Ω
- The set of all events is the ________________, denoted $\mathcal{F}$ or $\mathcal{A}$
- **EX: Flipping a coin**

  The possible events are:
  - Heads occurs
  - Tails occurs
  - Heads or Tails occurs
  - Neither Heads nor Tails occurs
- Thus, the event class can be written as

- **EX: Rolling a 6-sided die**

  There are many possible events, such as:

  1. The number 3 occurs
  2. An even number occurs
  3. No number occurs
  4. Any number occurs

- *Express the events listed in the example above in set notation.*

  1.

  2.

  3.

  4.

MORE ON EQUALLY-LIKELY OUTCOMES

IN DISCRETE SAMPLE SPACES

Consider the following examples:

A. A coin is tossed 3 times and the sequence of H and T is noted.
$\Omega_3 = \{HHH, HHT, HTH, \ldots, TTT\}$

Assuming equally likely outcomes,

$$P(a_i) = \frac{1}{|\Omega_3|} = \frac{1}{8} \text{ for any outcome } a_i$$

Let $A_2 =$ exactly 2 heads occurs in 3 tosses.
Then

Note that for equally likely outcomes in general, if an event E consists of K outcomes (i.e,. $E = \{o_1, o_2, \ldots, o_K\}$ and $|E| = K$), then

B. Suppose a coin is tossed 3 times, but only the number of heads is noted
$\Omega = \{0, 1, 2, 3\}$

Assuming equally likely outcomes,

$$P(a_i) = \frac{1}{|\Omega|} = \frac{1}{4} \text{ for any outcome } a_i$$

Then

Extension of Equally Likely Outcomes

to Continuous Sample Spaces

- Cannot directly apply our notion of equally-likely outcomes to continuous sample spaces because any continuous sample space has an infinite number of outcomes
- Thus, if each outcome has some positive probability of occurring $p > 0$, the total probability would be $\infty p = \infty$
- Our previous work on equally likely outcomes indicates that the probability of each outcome should be $1/\Omega = 0$, but this seems to suggest that the probability of every event is 0.
- In fact, assigning 0 probability to the outcomes does **NOT** necessarily imply that the probability of every event is 0.
- Let's use our previous approach of using cardinality to define the probability of an event. Consider the continuous sets which are simplest to measure cardinality on: the intervals

DEFN For a sample space Ω that is an interval, $\Omega = [A, B]$, a probability measure P has equally likely outcomes on Ω if for any a and b such that $A \le a \le b \le B$,

$$P([a, b]) \quad =$$

- Note that $P(c) = 0$ for any $c \in [A, B]$.
- For a typical continuous sample space, the prob. of any particular outcome is **zero**.

  Interpretation: Suppose we choose random numbers in $[A, B]$ until we see all the outcomes in that range at least once.

  How many trials will it take? ∞
- Seeing a number a finite number of times in an infinite number of trials $\Rightarrow$ Relative freq. = 0

  (Does not mean that outcome does not ever occur, only that it occurs very infrequently.)