EEL 5544 Noise in Linear Systems Lecture 33

RANDOM PROCESSES (RPS)

Example: Noise in a Linear System

- Want to define RPs in very similar ways as RVs
- **Recall** A RV is a _____________ that maps elements in Ω onto the real numbers

DEFN A (real-valued) random process is a _____________ that maps elements in Ω onto real-valued _______________:

- A RP can be denoted as $X(t;\omega)$.As for the case of random variables, we usually suppress the dependence of ω and just write $X(t)$
- The index of the function is usually taken to be t for time but can also represent space, etc.
- For a particular $\omega = \omega_0$, can draw $X(t;\omega_o)$
- $X(t;\omega_0)$ is a deterministic function and is called a _________________________ for $X(t;\omega)$
  Example

DEFN A *random process* or *stochastic process* on a probability space $(\Omega, \mathcal{F}, P)$ is an ______________________________ $\{X_t : t \in \mathbf{T}\}$ of ____________________________ on $(\Omega, \mathcal{F}, P)$.

- t typically denotes time with $\mathbf{T} \subset \mathbf{R}$
  - $\mathbf{T}$ can be discrete:
    - $\mathbf{T} = \mathbf{Z}$ (the integers)
    - $\mathbf{T} = \{0, 1, 2, 3, \ldots\}$
    - $\mathbf{T} = \left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\right\}$
  - or $\mathbf{T}$ can be continuous:
    - $\mathbf{T} = \mathbf{R}$
    - $\mathbf{T} = [0, \infty)$
    - $\mathbf{T} = (a, b)$
- RPs can also be classified according to the values they take on:

DEFN The ________________ $\mathcal{S}$ of a RP X is the ________________ (i.e., the set of ____________________________.

  - if $\mathcal{S}$ is a discrete set, then X is a ___________________________ RP.
  - if $\mathcal{S}$ is a continuous set, then X is a ________________________________ RP.

(The oscillator with random phase is an example of a continuous-time, continuous-amplitude random process.)

(The previous linear system example has inputs and outputs that are discrete-time, discrete-amplitude random process.)

Example

DISTRIBUTION AND DENSITY FUNCTIONS FOR RPS

- Recall that for each $t \in \mathbf{T}$, $X(t)$ is a RV
- Thus for each $t \in \mathbf{T}$, $\{X(t) \leq x\} \in \mathcal{A}$ *(is an event)*

$$\Rightarrow \bigcap_{k=1}^{n} \{X_{t_k} \leq x_k\} \in \mathcal{A}$$

$$\Rightarrow P\left[\bigcap_{k=1}^{n} \{X_{t_k} \leq x_k\}\right]$$

is defined.

$\Rightarrow$ We can define distribution functions for the random process

Let $X(t)$ be a random process on a probability space $(\Omega, \mathcal{F}, P)$.

DEFN The *one-dimensional cumulative distribution function* for $X(t)$ is given by

DEFN The *n-dimensional cumulative distribution function* for $X(t)$ is given by

$$\begin{aligned} &F_{X,n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) \\ &\quad = P\left[X(t_1) \leq x_1, X(t_2) \leq x_2, \right. \\ &\qquad\qquad \left. \ldots, X(t_n) \leq x_n\right] \\ &\quad = P\left[\bigcap_{k=1}^{n} \{X(t_k) \leq x_k\}\right] \end{aligned}$$

- For our purposes, a RP is completely specified by its n-dimensional cdfs for all positive integers n

DEFN The *n-dimensional probability density function* for a random process $X(t)$ is the function $f_{X,n}(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n)$ such that

$$F_{X,n}(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X,n}(u_1, ..., u_n; t_1, ..., t_n) \, du_n ... du_1.$$

DEFN The *n-dimensional probability mass function* for a discrete-valued random process $X(t)$ is given by

$$p_{X,n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P\left[X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n\right]$$

– Let the state space for $X(t)$ be given by

$$\mathcal{S} = \{s_k \,|k \in \mathbf{N}\},$$

where $s_i < s_j$ whenever $i < j$.

– Then

$$p_{X,1}(s_i; t) = F_{X,1}(s_i; t) - F_{X,1}(s_{i-1}; t),$$

and

$$\begin{aligned} p_{X,2}(s_i, s_j; t_1, t_2) &= F_{X,2}(s_i, s_j; t_1, t_2) \\ &\quad - F_{X,2}(s_{i-1}, s_j; t_1, t_2) \\ &\quad - F_{X,2}(s_i, s_{j-1}; t_1, t_2) \\ &\quad + F_{X,2}(s_{i-1}, s_{j-1}; t_1, t_2) \end{aligned}$$

Example