EEL 5544 Noise in Linear Systems Lecture 35

## PROPERTIES OF AUTOCORRELATION FUNCTIONS

- Given a 2nd-order RP $X(t)$, its autocorrelation functions satisfies:

(1) $R_X(t,t) \geq 0$ (positive power)

(2) $R_X(t_1,t_2) = R_X(t_2,t_1)$ *(symmetric)*

(3) $|R_X(t_1,t_2)| \leq \sqrt{R_X(t_1,t_1)R_X(t_2,t_2)}$

(4) $R_X$ is ______________________________

**DEFN** A function $f(t_1,t_2)$ defined on $\mathbf{T}$ is ______________________________ if for any $n \in \mathbf{Z}^+$, any $t_1, t_2, \ldots, t_n$ in $\mathbf{T}$, and any complex constants $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_i \alpha_k^* f(t_i,t_k) \geq 0$$

(both real and non-negative).

- Note 1: Property (4) is harder to prove than (1)-(3).In this class, we will usually only test property (4) by testing an equivalent condition in the frequency domain.
- Note 2: property (4) implies properties (1)-(3).
- Note 3: autocovariance functions also satisfy properties (1)-(4):

  Pf: consider $Y(t) = X(t) - \mu_X(t)$

  Then $R_Y(t_1,t_2) = C_X(t_1,t_2)$
- Note 4: Any symmetric, nonnegative definite function $C(t_1,t_2)$ is an autocovariance function of some 2nd-order RP

- Note 5: Gaussian RPs are completely specified by $\mu_X(t)$ and $\text{Cov}\left\{X(t_1), X(t_2)\right\}$, so only need to know mean and autocovariance functions

  **Examples**

## STATIONARY RANDOM PROCESSES

**DEFN** A RP $X(t)$ is *strict-sense stationary* (stationary or strictly stationary) if

$$F_{X,n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) \\ = F_{X,n}(x_1, x_2, \ldots, x_n; t_1+t_0, t_2+t_0, \\ \ldots, t_n + t_0)$$

for each $n \in \mathbf{Z}^+$, each choice of $t_1, t_2, \ldots, t_n$ in $\mathbf{T}$ and all $t_0 \in \mathbf{T}$.

- abbreviate *strict-sense stationary* by *SSS*
- also called *shift-invariant*
- Note: the definition requires that $\mathbf{T}$ be closed under addition (i.e., if $a, b \in \mathbf{T}$ then $a+b \in \mathbf{T}$.)
  (This is true for many sets of interest:$\mathbf{R}$, $[0, \infty)$, $\mathbf{Z}$, $\mathbf{Z}^+$)

  **Examples**

## CONSEQUENCES OF STATIONARITY

- Consider $X(t)$ for fixed $t = t_0$
- Then $X(t_0)$ is a RV
- The mean $\mu_X(t_0) = E[X(t_0)]$ depends only on $f_{X,1}(x; t_0)$
- By defn of SSS, $f_{X,1}(x; t_0) = f_{X,1}(x; t_0 + \tau)$ for any $\tau \in \mathbb{T}$

  $\Rightarrow f_{X,1}(x; t_0)$ does not depend on $t_0$

  $\Rightarrow \mu_X(t_0)$ does not depend on $t_0$

  $\Rightarrow \mu_x(t) = \mu_X$ (a constant)
- If $X(t)$ is SSS, $f_{X,2}(x_1, x_2; t + \tau, t) = f_{X,2}(x_1, x_2; t_0 + \tau, t_0)$ (i.e., does not depend on $t$, but may depend on $\tau$

  $\Rightarrow R_X(t + \tau, t)$ does not depend on $t$, but may depend on $\tau$

  For convenience, we write $R_X(\tau)$ in this case
- For the covariance function of a SSS RP,

$$
\begin{aligned}
C_X(t+\tau, t) &= R_X(t+\tau, t) - \mu_X(t+\tau)\mu_X(t) \\
&= \mathbb{R}_X(\tau) - \mu_X^2 \\
&= C_X(\tau)
\end{aligned}
$$

**Summary**

$$
\begin{aligned}
\mu_X(t) &= \mu_X && (\mathbf{S1}) \\
R_X(t+\tau, t) &= R_X(\tau) && (\mathbf{S2}) \\
C_X(t+\tau, t) &= C_X(\tau) && (\mathbf{S3})
\end{aligned}
$$

We can use these properties to define weaker types of stationarity:

**DEFN** A 2nd-order RP $X(t)$ is ____________________ if **(S1)** and **(S2)** hold for all $t$ and $\tau$.

**DEFN** A 2nd-order RP $X(t)$ is ____________________ if **(S3)** holds for all $t$ and $\tau$.

**Notes**

1. For $X(t)$ WSS, $C_x(t+\tau, t) = R_X(\tau) - \mu_X^2$, so $X(t)$ is also CSS ____________________________
2. Does CSS $\Rightarrow$ WSS?

   **Example**

## PROPERTIES OF AUTOCORRELATION FUNCTION FOR WIDE-SENSE RPS

For real WSS RPs, the three properties of the autocorrelation function simplify to:

I. $R_X(0) = E[X^2(t)] \geq 0$ (____________________________)

II.

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] \\ &= E[X(t_1-\tau)X(t_1)], \text{ where } t_1 = t+\tau \\ &= R_X(-\tau) \text{ (\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)} \end{aligned}$$

III. $|R_X(\tau)| \leq R_X(0)$ (__________________________________)

*Proof:* From previous class,

$$\begin{aligned} |R_X(t_1, t_2)| &\leq \frac{R_X(t_1, t_1) + R_X(t_2, t_2)}{2} \\ &= \frac{R_X(0) + R_X(0)}{2} \end{aligned}$$

**Example**