EEL 5544 Noise in Linear Systems Lecture 6

BY REQUEST: THE GAME SHOW/MONTY HALL PROBLEM

Slightly paraphrased from the Ask Marilyn column of Parade magazine:

- Suppose you're on a game show, and you're given the choice of three doors:
  - Behind one door is a car
  - Behind the other doors are goats
- You pick a door, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.
- The host then offers you the option to switch doors? Does it matter if you switch?
- Compute the probabilities of winning for the following three strategies:
  1. Never switch
  2. Always switch
  3. Flip a fair coin, and switch if it comes up heads

SEQUENTIAL EXPERIMENTS

DEFN A *sequential experiment* (or *combined experiment*) consists of a series of subexperiments.

Prob Space for N Sequential Experiments

1. The combined sample space:$\Omega =$ __________________, the ________________________ of the sample spaces for the subexperiments
   - Then any $A \in \Omega$ is of the form $(B_1, B_2, \ldots, B_N)$, where $B_i \in \Omega_i$
2. Event class $\mathcal{F}$:σ-algebra on Ω
3. $P(\cdot)$ on $\mathcal{F}$ such that

$$P(A) \quad =$$

For s.i. subexperiments,

$$P(A) \quad =$$

Bernoulli Trials

DEFN A **Bernoulli trial** is an experiment that is performed once with outcomes of *success* or *failure*.

- Let p denote the probability of success
- Then for n independent Bernoulli trials, the prob. of a *specific* arrangement of k successes (and $n - k$ failures) is ______
- The number of ways that k success can occur in n places is ______
- Thus, the probability of k successes on n trials is

$$p_n(k) = \tag{1}$$

Examples

- When n is large, the probability in (1) can be difficult to compute, as $\binom{n}{k}$ gets very large at the same time that one or both of the other terms gets very small
- For large n, we can use a Gaussian approximation to approximate binomial probabilities. Details will be provided in a supplementary lecture to be posted to E-learning

Geometric Probability Law

Suppose we repeat independent Bernoulli trials until the first success.
What is the prob. that m trials are required?

$$\begin{aligned} p(m) &= P\left(\{m \text{ trials to first success}\}\right) \\ &= P\left(\{\text{1st } m-1 \text{ trials are failures}\} \cap \right. \\ &\quad \left. \{m\text{th trial is success}\}\right) \end{aligned}$$

Then

$$p(m) \quad =$$

Also, $P\left(\{\# \text{ trials} > k\}\right) =$ ____________

Example

THE POISSON LAW

Example: A mobile base station receives an average of 30 calls per hour during a certain time of day.
Construct a probabilistic model for this.

- How should we proceed? What should we assume?
- Let's start with a binomial model.
- *Attempt 1:* If we assume that only one call can come in during any 1-minute period, then we can model the call arrivals by Bernoulli experiments with probability of success $p = 0.5$
- Then an average of 30 calls come in per hour, with a maximum of 60 calls/hour.
- However, the limit on only one call per 1-minute period seems artificial, as does the limit of a maximum of 120 calls/hour.
- *Attempt 2:* Let's assume that only one call can come in during any 30 second period. Then we can model the call arrivals by Bernoulli experiments with probability of success $p = 0.25$.
- Again, we have an average of 30 calls/hour. However, now calls can come in at 30-second intervals. There are now 120 experiments, so the maximum possible number of calls/hour is 120.
- *Attempt n:* Let's assume that calls come in during $1/n$th of a minute. For there to be an average of 30 calls/hour, then $p = 1/(2n)$, so $np = 0.5$ is a constant. Then the maximum possible number of calls/hour is n.
- As $n \to \infty$, the number of subintervals goes to infinity at the same rate as the probability of an arrival in any particular interval goes to zero.
- For finite k, the probability of k arrivals is a Binomial probability

$$\binom{n}{k} p^k (1-p)^{n-k}$$

- If we let $np = a$ be a constant, then for large n,

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{k!} a^k \left(1 - \frac{a}{n}\right)^{n-k}$$

- Then

$$\lim_{n\to\infty} \frac{1}{k!} a^k \left(1 - \frac{a}{n}\right)^{n-k} = \frac{a^k}{k!} e^{-a},$$

  which is the Poisson law

- The Poisson law gives the probability for events that occur randomly in space or time

- Examples are:
  - calls coming in to a switching center
  - packets arriving at a queue in a network
  - processes being submitted to a scheduler

    The following examples are adopted from A First Course in Probability by Sheldon Ross:
  - # of misprints on a group of pages in a book
  - # of people in a community that live to be 100 years old
  - # of wrong telephone numbers that are dialed in a day
  - # of α-particles discharged in a fixed period of time from some radioactive material
  - # of earthquakes per year
  - # of computer crashes in a lab in a week

  The Poisson RV can be used to model all of these phenomena!

- Let λ= the # of events/(unit of space or time)
- Consider observing some period of time or space of length t and let $\alpha = \lambda t$
- Let N= the # events in time (or space) t
- Then

$$P(N = k) = \begin{cases} \frac{\alpha^k}{k!}e^{-\alpha}, & k = 0, 1, \ldots \\ 0, & o.w. \end{cases}$$

EX: Phone calls arriving at a switching office

If a switching center takes 90 calls/hr, what is the prob. that it receives 20 calls in a 10 minute interval?

$\lambda = 90/hr$, $t = 1/6\ hr$
$\alpha = \lambda t = 15$

$$P(N = 20) = \frac{15^{20}}{20!}e^{-15} \approx 4.2 \times 10^{-2}$$

- The Gaussian approximation can also be applied to Poisson probabilities – see pages 46 and 47 in the book for details