EEL 5544 Lecture 8

PROBABILITY DENSITY FUNCTION

DEFN The ________________________________ (______), $f_X(x)$, of a random variable X is the derivative (which may not exist at some places) of ____:

- Properties:

1. $f_X(x) \geq 0, -\infty < x < \infty$

2. $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$

3. $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

4. $P(a < X \leq b) = \int_{a}^{b} f_X(x)dx, a \leq b$

5. If $g(x)$ is a nonnegative piecewise continuous function with finite integral

$$\int_{-\infty}^{+\infty} g(x)dx = c, -\infty < c < +\infty,$$

then $f_X(x) = \frac{g(x)}{c}$ is a valid pdf.

Pf omitted.

- Note that if $f(x)$ exists at x, then $F_X(x)$ is continuous at x and thus $P[X = x] = F(x) - F(x^-) = 0$
- Recall that this does not mean that x never occurs, but that the occurrence is extremely unlikely

DEFN If $F_X(x)$ is continuous for every x and its derivative exists everywhere except at a countable set of points, then X is a ________________________________.

- Where $F_X(x)$ is continuous but $F'_X(x)$ is discontinuous, any positive number can be assigned to $f_X(x)$, and thus $f_X(x)$ will be assigned to every point of $f_X(x)$ when X is a continuous random variable

Uniform Continuous Random Variables

DEFN For a *uniform RV* on an interval $[a, b]$, any two subintervals of $[a, b]$ that have the same length will have equal probabilities. The density function is given by

$$f_X(x) = \begin{cases} 0, & x < a \\ \frac{1}{b-a}, & a \leq x \leq b \\ 0, & x > b. \end{cases}$$

DEFN A ____________________________________ has probability concentrated at a countable number of values.It has a staircase type of distribution function.

PROBABILITY MASS FUNCTION

DEFN For a discrete RV, the *probability mass function* (pmf) is

EX: Roll a fair 6-sided die

X= # on top face

$$P(X = x) = \begin{cases} 1/6, & x = 1, 2, \ldots, 6 \\ 0, & o.w. \end{cases}$$

EX: Flip a fair coin until heads occurs

X = # of flips

$$P(X = x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, \ldots \\ 0, & o.w. \end{cases}$$

IMPORTANT RANDOM VARIABLES

Discrete RVs

1. Bernoulli RV

   - An event $A \in \mathcal{A}$ is considered a "success"
   - A Bernoulli RV X is defined by

$$X = \begin{cases} 1, & s \in A \\ 0, & s \notin A \end{cases}$$

   - The pmf for a Bernoulli RV X can be found formally as

$$\begin{aligned} P(X=1) &= P\Big(X(s)=1\Big) \\ &= P\Big(\{s|s \in A\}\Big) = P(A) \triangleq p \end{aligned}$$

   So,

$$P(X=x) = \begin{cases} p & x=1 \\ 1-p & x=0 \\ 0 & x \neq 0,1 \end{cases}$$

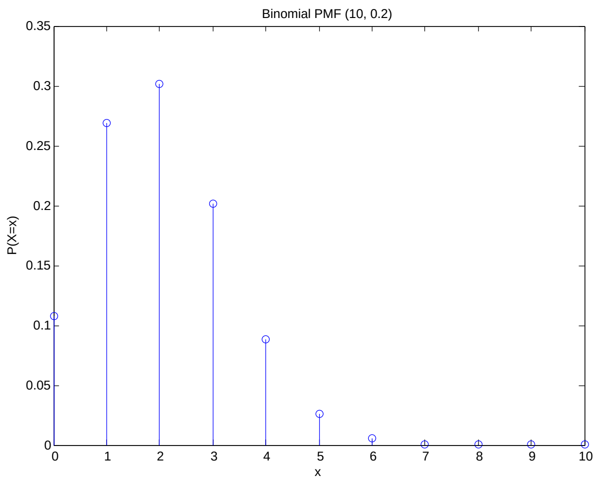
2. Binomial RV

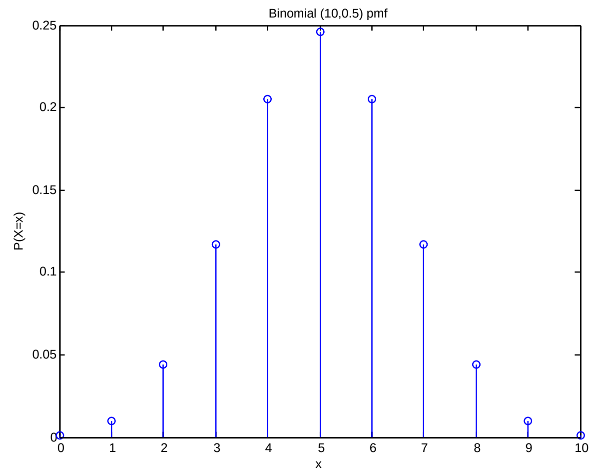
   - A Binomial RV represents the number of sucesses on n independent Bernoulli trials
   - Thus, a Binomial RV can also be defined as the sum of n independent Bernoullis RVs
   - Let X= # of successes

   Then the pmf of X is given by

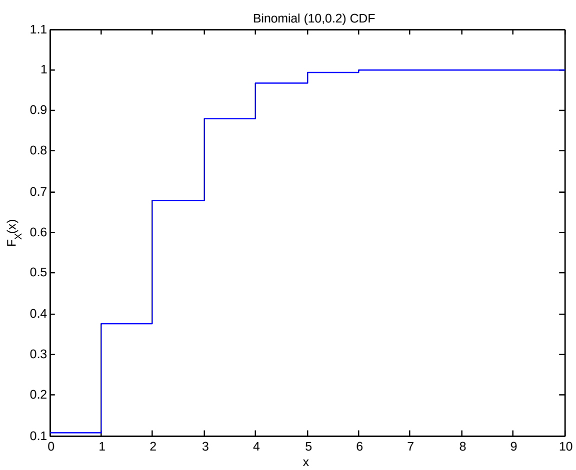
$$P[X=k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

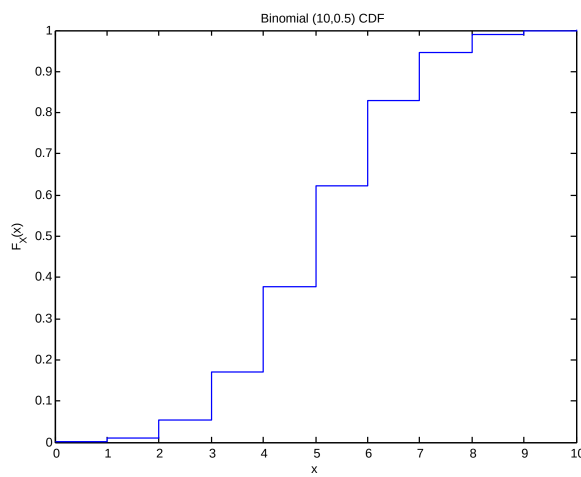
- The pmfs for two Binomial RVs with $n = 10$ and $p = 0.2$ or $p = 0.5$ are shown below



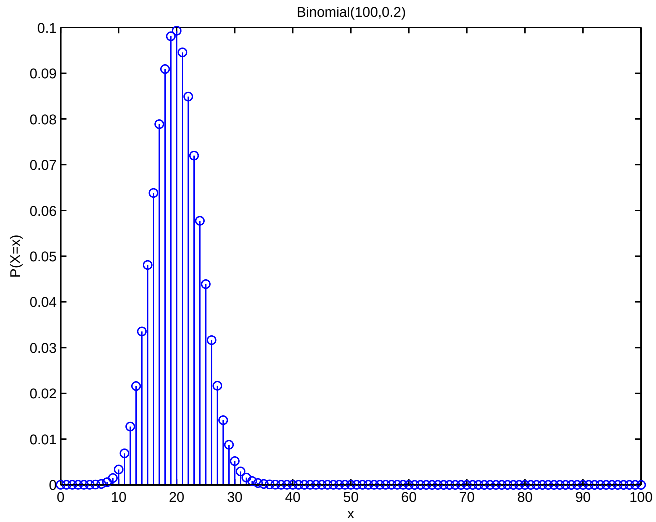


- The cdfs for two Binomial RVs with $n = 10$ and $p = 0.2$ or $p = 0.5$ are shown below





- When n is large, the Binomial pmf has a bell shape. For example, the pmf below is for $n = 100$ and $p = 0.2$



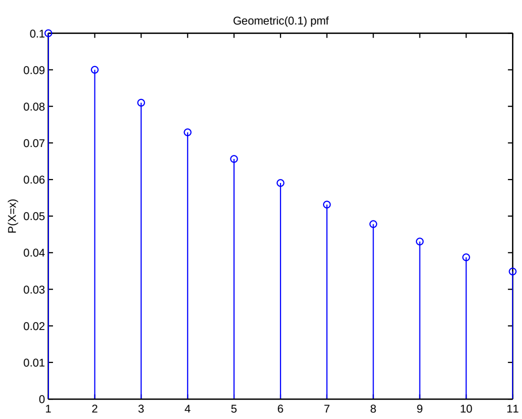
3. <u>Geometric RV</u>

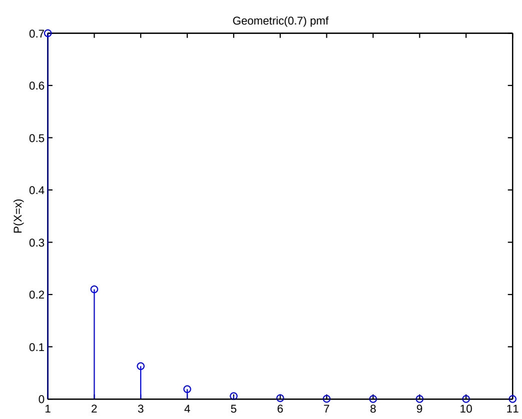
- A Geometric RV occurs when independent Bernoulli trials are conducted until the first success
- X = # number of trials required

  Then the pmf or X is given by

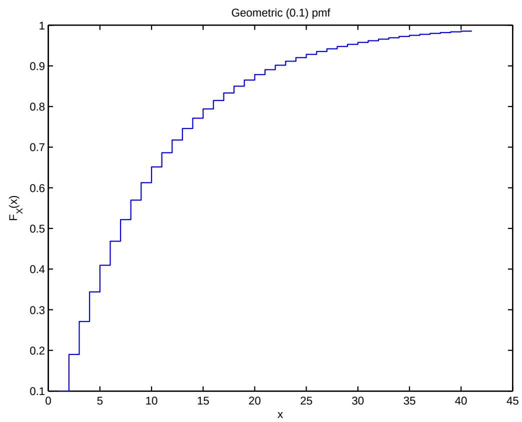
$$P[X = k] = \begin{cases} (1-p)^{k-1}p, & k = 1, 2, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

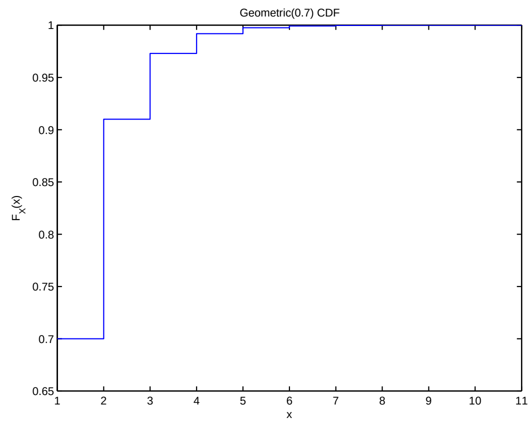
- The pmfs for two Geometric RVs $p = 0.1$ or $p = 0.7$ are shown below





- The cdfs for two Geometric RVs with $p = 0.1$ or $p = 0.7$ are shown below





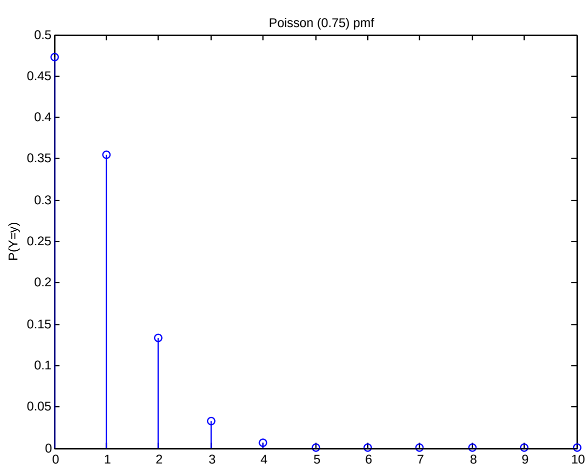
4. Poisson RV

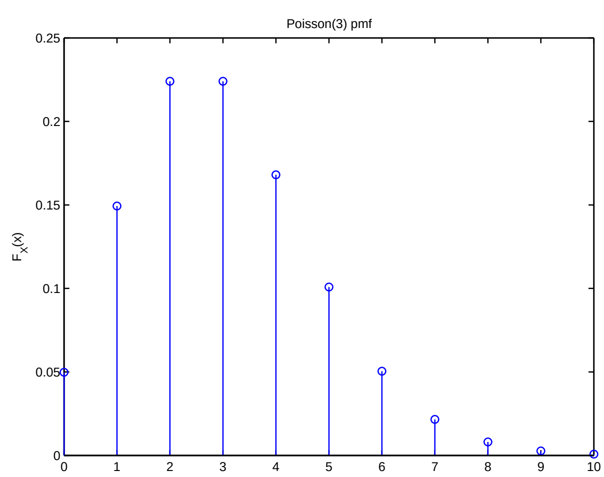
- Models events that occur randomly in space or time

- Let λ= the # of events/(unit of space or time)
- Consider observing some period of time or space of length t and let $\alpha = \lambda t$
- Let N= the # events in time (or space) t
- Then

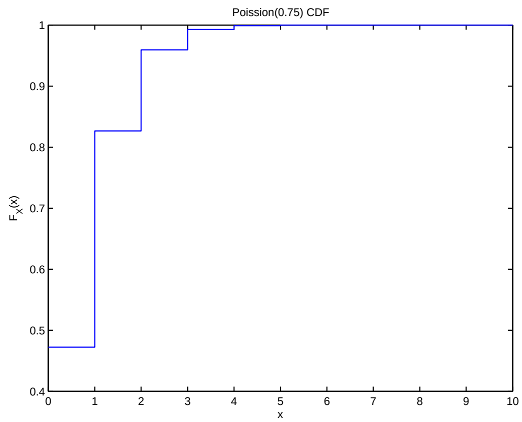
$$P(N = k) = \begin{cases} \frac{\alpha^k}{k!} e^{-\alpha}, & k = 0, 1, \ldots \\ 0, & \text{o.w.} \end{cases}$$

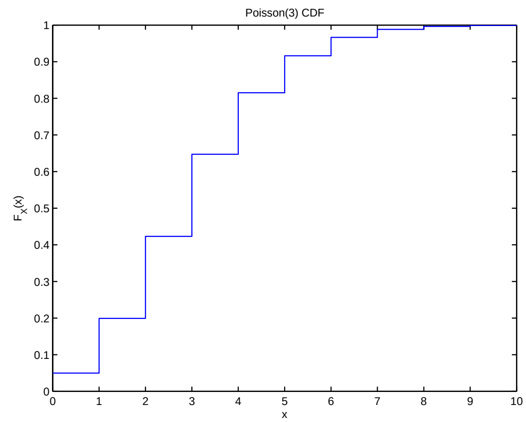
- The pmfs for two Poission RVs with $\alpha = 0.75$ or $\alpha = 3$ are shown below



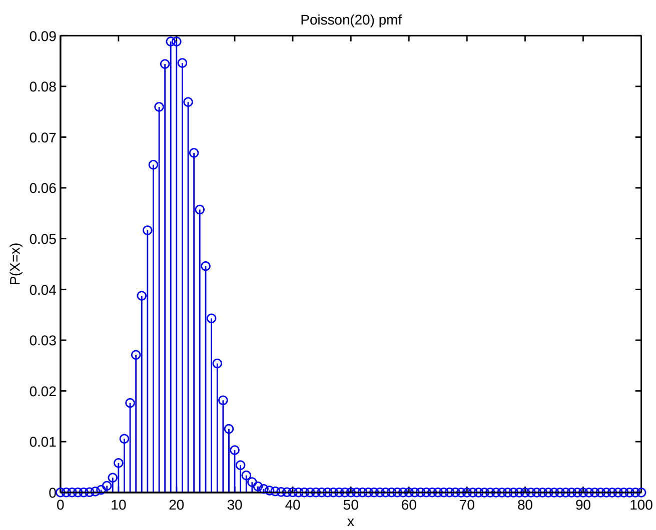


- The cdfs for two Poission RVs with $\alpha = 0.75$ or $\alpha = 3$ are shown below





- For large α, the Poisson pmf has a bell shape. For example, the pmf below if for a Poisson RV with $\alpha = 20$:



Important Continuous RVs

1. Uniform RV:covered in example
2. Exponential RV
   - Characterized by single parameter λ

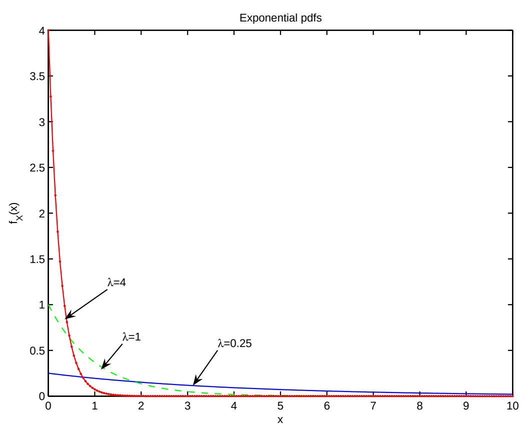
- Density function:

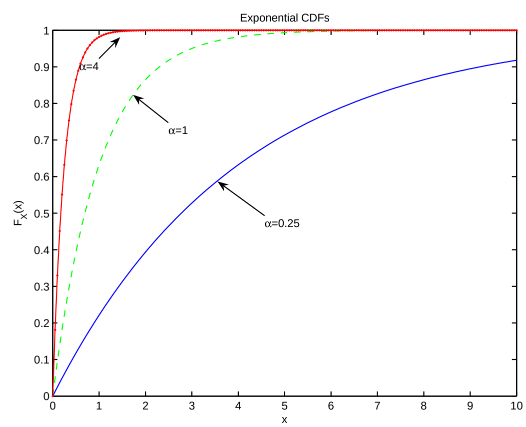
$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

- Distribution function:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

- The book's definition substitutes $\lambda = 1/\mu$
- Obtainable as limit of geometric RV
- The pdf and CDF for exponential random variables with λ equal to 0.25, 1, or 4 are shown below





3. <u>Gaussian RV</u>

- For large n, the binomial and Poisson RVs have a bell shape
- Consider the binomial:
- DeMoivre-Laplace Theorem: If n is large and p is small, then if $q = 1 - p$,

$$\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} \exp\left\{-\frac{(k-np)^2}{2npq}\right\}$$

- Let $\sigma^2 = npq$, $\mu = np$, then

$$\begin{aligned} P(\quad k \quad & \text{successes on } n \text{ Bernoulli trials}) \\ &= \binom{n}{k} p^k q^{n-k} \\ &\approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(k-\mu)^2}{2\sigma^2}\right\}. \end{aligned}$$

DEFN A *Gaussian random variable* X has density function

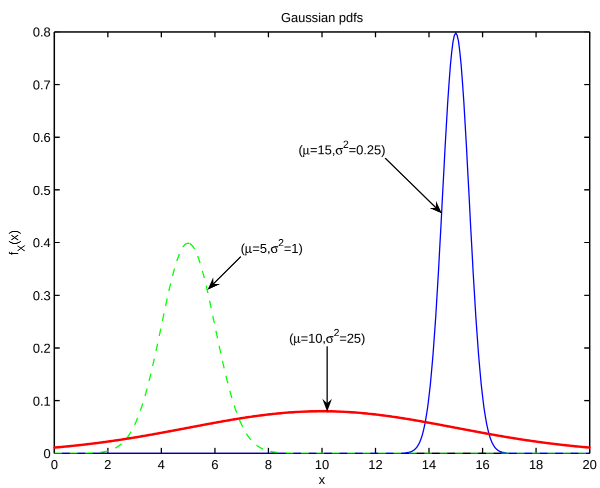
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

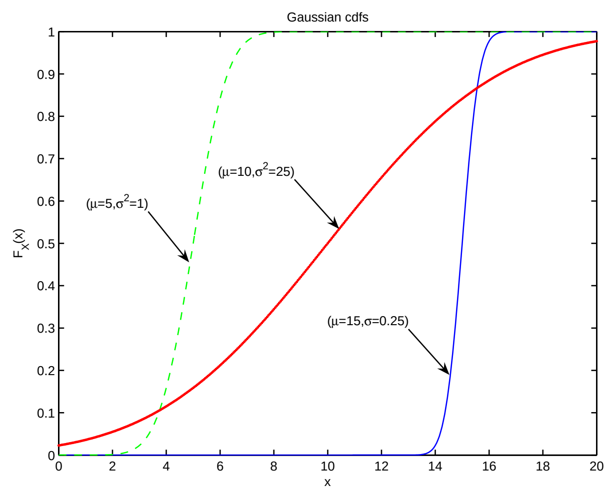
with parameters (mean) μ and (variance) $\sigma^2 \geq 0$.

- *In fact, the sum of a large number of almost any type of independent random variables converges (in distribution) to a Gaussian random variable (Central Limit Theorem).*
- The CDF of a Gaussian RV is given by

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\} dt, \end{aligned}$$

  which cannot be evaluated in closed form.
- The pdf and CDFs for Gaussian random variables with $(\mu = 15, \sigma^2 = 0.25)$, $(\mu = 5, \sigma^2 = 1)$, and $(\mu = 10, \sigma^2 = 25)$





- Instead, we tabulate distribution functions for a normalized Gaussian variable with $\mu = 0$ and $\sigma^2 = 1$:
  - The CDF for the normalized Gaussian RV is defined as

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} dt.$$

  - Engineers more commonly use the complementary distribution function, or Q-function, defined by

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} dt.$$

- Note that $Q(x) = 1 - \Phi(x)$
- I will be supplying you with a Q-function table and a list of approximations to $Q(x)$
- The Q-function can also be defined as

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left\{ -\frac{x^2}{\sin^2 \phi} \right\} d\phi.$$

  (This is a fairly recent result that is in very few textbooks. This form has a finite range of integration that is often easier to work with.)

- The CDF for a Gaussian RV with mean μ and variance σ^2 is

$$\begin{aligned} F_X(x) &= \Phi\left(\frac{x-\mu}{\sigma}\right) \\ &= 1 - Q\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

  Note that the denominator above is σ, not σ^2. Many students use the wrong value when working tests!

- To find the prob. of some interval using the Q-function, it is easiest to rewrite the prob:

$$\begin{aligned} P(a < X \leq b) &= P(X > a) - P(X > b) \\ &= Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right) \end{aligned}$$

Examples on board