1.1.6 SINGULARITY FUNCTIONS

CT Impulse Function

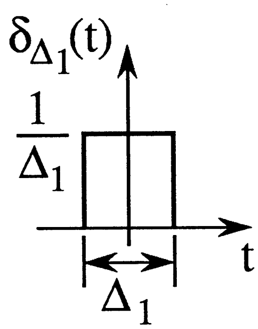
Let $\delta_\Delta(t) = \dfrac{1}{\Delta}\text{rect}\left(\dfrac{t}{\Delta}\right)$

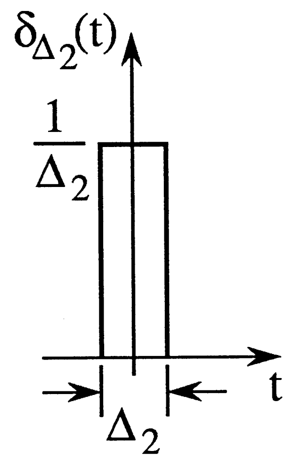
Note that $\int_{-\infty}^{\infty} \delta_\Delta(t)dt = 1$

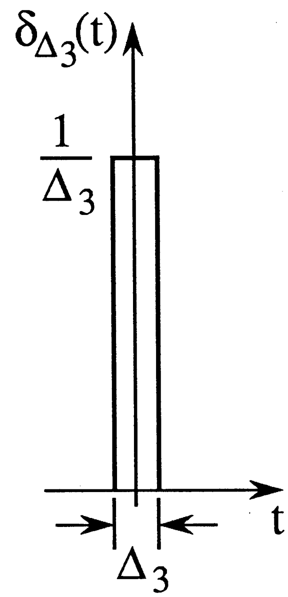
What happens

as $\Delta \to 0$?

$\Delta_1 > \Delta_2 > \Delta_3$







In the limit, we obtain

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$= \begin{cases} 0\,, & t \neq 0 \\ \infty\,, & t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

We will use the impulse in three ways:

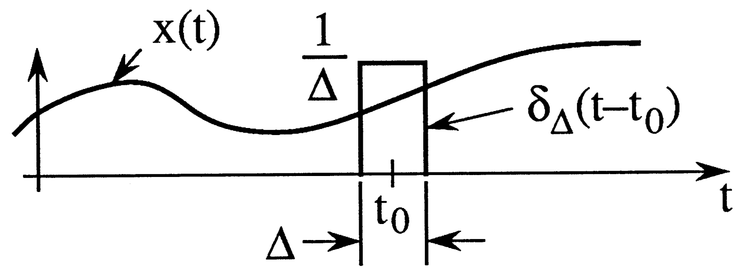
1. to sample signals

2. as a way to decompose signals into elementary components

3. as a way to characterize the response of a class of systems

Sifting Property

Consider a signal x(t) multiplied by $\delta_\Delta(t - t_0)$ for some fixed t_0:



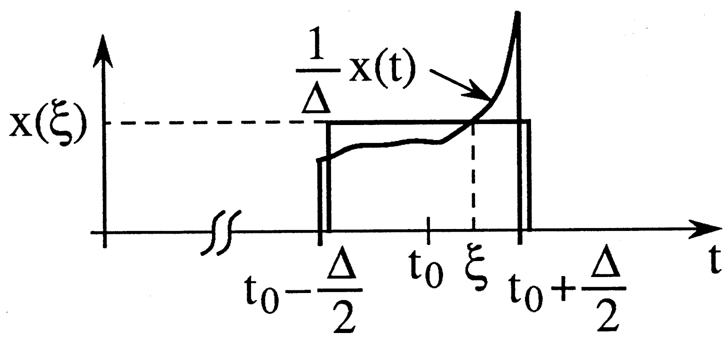
Let's integrate over the entire real line.

From the mean-value theorem of calculus,

$$\int_{-\infty}^{\infty} x(t)\delta_{\Delta}(t - t_0)dt = x(\xi)$$

for some ξ which satisfies

$$t_0 - \frac{\Delta}{2} \leq \xi \leq t_0 + \frac{\Delta}{2}$$



As $\Delta \rightarrow 0$, $\xi \rightarrow t_0$ and $x(\xi) \rightarrow x(t_0)$

So we have

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

provided $x(t)$ is continuous at t_0

Equivalence

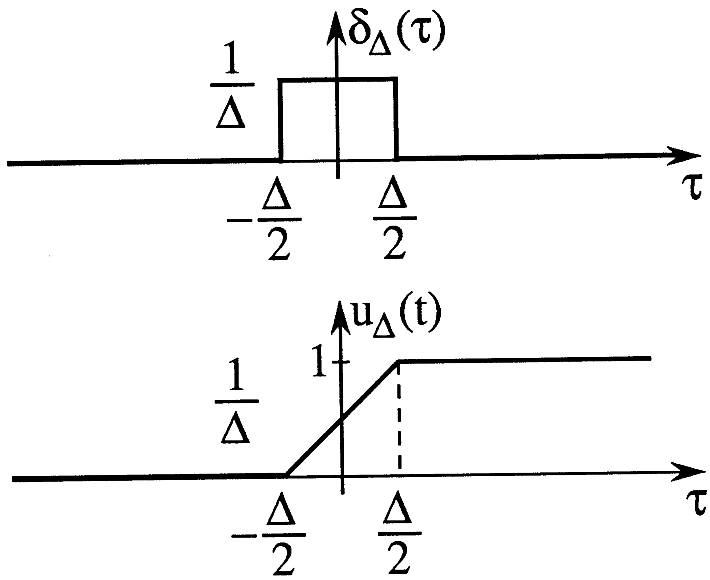
Based on sifting property,

$$x(t)\delta(t - t_0) \equiv x(t_0)\delta(t - t_0) \ .$$

These two expressions may be used interchangeably.

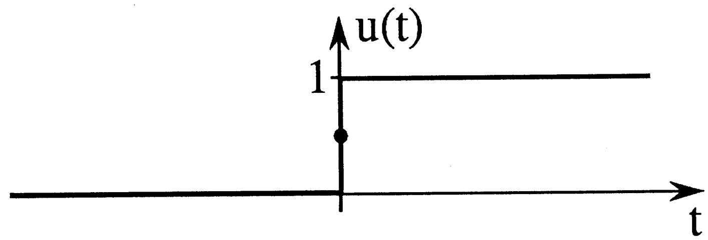
Indefinite Integral of $\delta(t)$

Let $u_\Delta(t) = \int_{-\infty}^{t} \delta_\Delta(\tau)\, d\tau$



Let $\Delta \rightarrow 0$,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \, .$$



When we defined the unit step function, we did not specify its value at t = 0. In this case, u(0) = 0.5.

Why can't we apply sifting property to

$$\int_{-\infty}^{t} \delta(\tau)d\tau \ ?$$

definite integrals vs. indefinite integrals

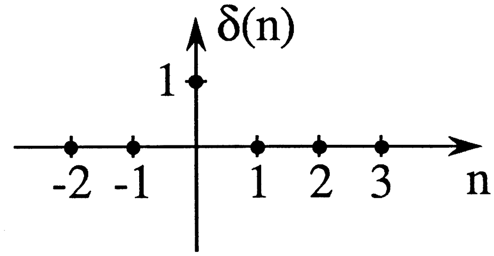
More general form of sifting property

$$\int_{a}^{b} x(\tau)\delta(\tau - t_0)d\tau = \begin{cases} x(t_0)\,, & a < t_0 < b \\ 0\,, & t_0 < a \;\text{ or }\; b < t_0 \end{cases}$$

DT Impulse Function (unit sample function)

Much simpler than CT case, no limiting process is required.

$$\delta(n) = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & \text{else} \end{cases}$$



Sifting Property

$$\sum_{n=n_1}^{n_2} x(n)\,\delta(n - n_0) = \begin{cases} x(n_0) \ , & n_1 \le n_0 \le n_2 \\ 0 \ , & \text{else} \end{cases}$$

Equivalence

$$x(n)\ \delta(n - n_0) = x(n_0)\ \delta(n - n_0)$$

Indefinite Sum of $\boldsymbol{\delta}$(n)

$$u(n) = \sum_{m=-\infty}^{n} \delta(m)$$