1.2.1 SYSTEM PROPERTIES

Notation:

$$y(t) = \underset{\substack{\uparrow \\ \text{system}}}{S}[x(t)]$$

A. Linearity

def. A system S is *linear* (L) if for any two inputs $x_1(t)$ and $x_2(t)$ and any two constants a_1 and a_2, it satisfies:

$$S[a_1 x_1(t) + a_2 x_2(t)] = a_1 S[x_1(t)] + a_2 S[x_2(t)]$$

Special cases:

a. homogeneity (let $a_2 = 0$)

$$S[a_1 x_1(t)] = a_1 S[x_1(t)]$$

b. superposition (let $a_1 = a_2 = 1$)

$$S[x_1(t) + x_2(t)] = S[x_1(t)] + S[x_2(t)]$$

Examples

1. $$y(t) = \int_{t-1/2}^{t+1/2} x(\tau)d\tau$$

let $x_3(t) = a_1x_1(t) + a_2x_2(t)$

$$y_3(t) = \int_{t-1/2}^{t+1/2} x_3(\tau)d\tau$$

$$= \int_{t-1/2}^{t+1/2} [a_1x_1(\tau) + a_2x_2(\tau)]d\tau$$

$$y_3(t) = a_1 \int_{t-1/2}^{t+1/2} x_1(\tau)d\tau + a_2 \int_{t-1/2}^{t+1/2} x_2(\tau)d\tau$$

$$= a_1 y_1(t) + a_2 y_2(t)$$

$\therefore$ system is linear.

Can similarly show that

2. $y(n) = \frac{1}{3}\,[x(n) + x(n-1) + x(n-2)]$ is linear.

Consider two additional examples:

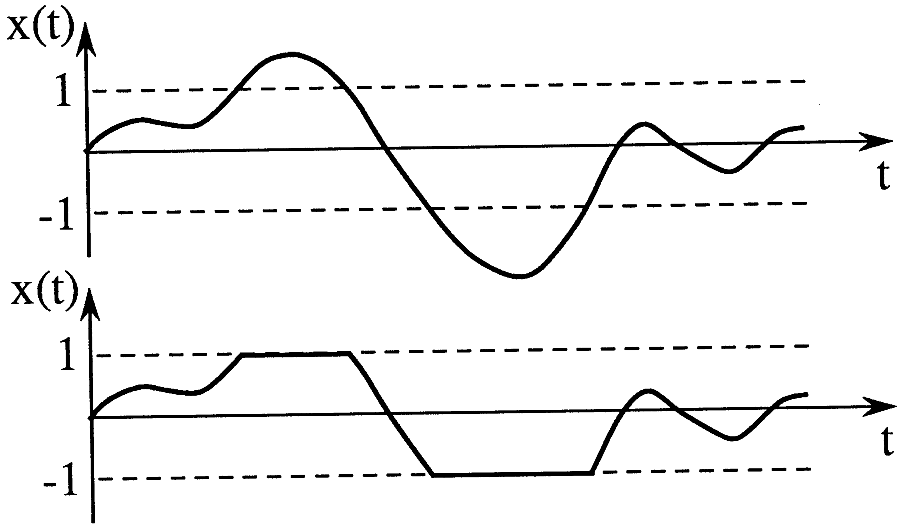
3. $y(n) = n\, x(n)$

let $x_3(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\begin{aligned} y_3(n) &= n\, x_3(n) \\ &= a_1\, n\, x_1(n) + a_2\, n\, x_2(n) \\ &= a_1\, y_1(n) + a_2\, y_2(n) \end{aligned}$$

$\therefore$ system is linear.

4. $$y(t) = \begin{cases} -1\,, & x(t) < -1\,, \\ x(t)\,, & -1 \le x(t) \le 1\,, \\ 1\,, & 1 < x(t)\,, \end{cases}$$



Suspect that system is nonlinear - find a counterexample.

$$x_1(t) \equiv 1 \quad \Rightarrow \quad y_1(t) \equiv 1$$

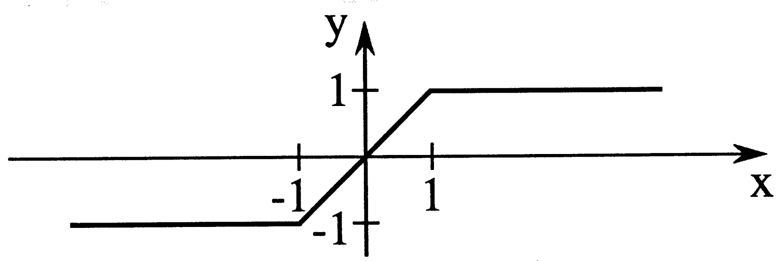
$$x_2(t) \equiv \frac{1}{2} \quad \Rightarrow \quad y_2(t) \equiv \frac{1}{2}$$

$$x_3(t) = x_1(t) + x_2(t) = \frac{3}{2}$$

$$\Rightarrow y_3(t) \equiv 1 \neq \frac{3}{2}$$

$\therefore$ system is not linear.

Because it is *memoryless,* this system is completely described by a curve relating input to output at each time t:



Importance of linearity: can represent response to a complex input in terms of responses to very simple inputs.

B. Time-invariance

def. A system S is *time-invariant* (TI) if delaying the input results in only an identical delay in the output, *i.e.*

$$\text{if} \quad y_1(t) = S[x_1(t)]$$

$$\text{and} \quad y_2(t) = S[x_1(t - t_0)]$$

$$\text{then} \quad y_2(t) = y_1(t - t_0)$$

Examples

2. $y(n) = \frac{1}{3}\,[x(n) + x(n-1) + x(n-2)]$

assume

$$y_1(n) = \frac{1}{3}\,[x_1(n) + x_1(n-1) + x_1(n-2)]$$

let $x_2(n) = x_1(n-n_0)$

$$\begin{aligned}
y_2(n) &= \frac{1}{3}\,[x_2(n) + x_2(n-1) + x_2(n-2)] \\
&= \frac{1}{3}\,[x_1(n-n_0) + x_1(n-1-n_0) + x_1(n-2-n_0)] \\
&= \frac{1}{3}\,[x_1(n-n_0) + x_1(n-n_0-1) + x_1(n-n_0-2)] \\
&= y_1(n-n_0)
\end{aligned}$$

$\therefore$ system is TI.

Can similarly show that

1. $y(t) = \int_{t-1/2}^{t+1/2} x(\tau)d\tau$ is TI

3. $y(n) = n\, x(n)$

assume $y_1(n) = n\, x_1(n)$

let $x_2(n) = x_1(n-n_0)$

$$
\begin{aligned}
y_2(n) &= n\, x_2(n) \\
&= n\, x_1(n-n_0) \\
&\neq (n-n_0)\, x_1(n-n_0)
\end{aligned}
$$

$\therefore$ system is not TI

To demonstrate, consider response to impulse:

$$
\begin{aligned}
x_1(n) &= \delta(n) &&\Rightarrow\ y_1(n) \equiv 0 \\
x_2(n) &= \delta(n-1) &&\Rightarrow\ y_2(n) = \delta(n-1) \\
& && \qquad\qquad\ \ \neq y_1(n-1)
\end{aligned}
$$

C. Causality

def. A system S is ***causal*** if the output at time t depends only on $x(\tau)$ for $\tau \leq t$.

Causality is equivalent to the property:

$$\text{If } x_1(t) = x_2(t),\ t \leq t_0,\ \text{then } y_1(t) = y_2(t),\ t \leq t_0$$

Examples

1. not causal
2. causal
3. causal
4. causal

D. Stability

def. A system is said to be ***bounded-input-bounded-output (BIBO) stable*** if every bounded input produces a bounded output, *i.e.* $M_x < \infty \Rightarrow M_y < \infty$.

Examples

2. $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$

Assume $|x(n)| \leq M_x$ for all n.

$$
\begin{aligned}
|y(n)| &= \frac{1}{3}\,|x(n) + x(n-1) + x(n-2)| \\
&\leq \frac{1}{3}\,[\,|x(n)| + |x(n-1)| + |x(n-2)|\,] \\
&\leq M_x
\end{aligned}
$$

1. It can similarly be shown that the system of Example 1 is also BIBO stable.

3. $y(n) = n\,x(n)$

   Let $x(n) \equiv 1$.
   Given any real M_y, $\exists$ N (namely $N = \lceil M_y \rceil$) such that $|y(n)| \geq M_y$ for $n \geq N$.

4. Since $|y(t)| \leq 1$, system is trivially BIBO stable. In fact, input can be unbounded.