1.2.2 CONVOLUTION

Characterization of behavior of any LTI system.

1. Decompose input into sum of simple basis functions $x_i(n)$.

$$x(n) = \sum_{i=0}^{N-1} c_i x_i(n)$$

$$y(n) = S\,[x(n)]$$

$$= \sum_{i=0}^{N-1} c_i\, S[x_i(n)]$$

2. Choose basis functions which are all shifted versions of a single basis function $x_0(n)$.

i.e. $x_i(n) = x_0(n - n_i)$

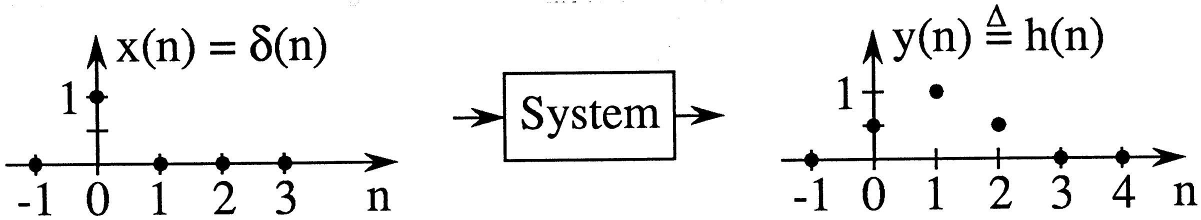
Let $y_i(n) = S\,[x_i(n)]$, $\quad i = 0, \ldots, N-1$

then $y_i(n) = y_0(n - n_i)$

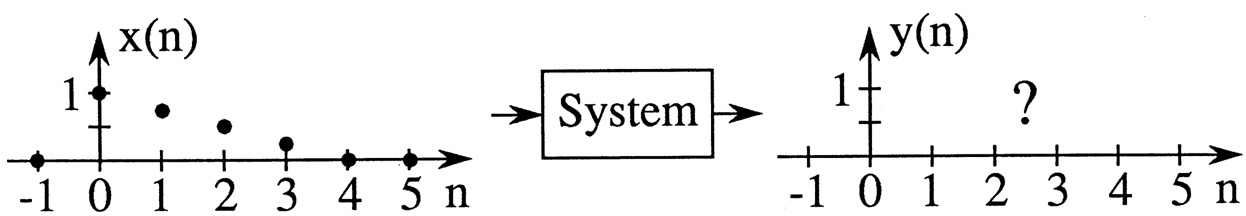
and $$y(n) = \sum_{i=0}^{N-1} c_i\, y_0(n-n_i).$$

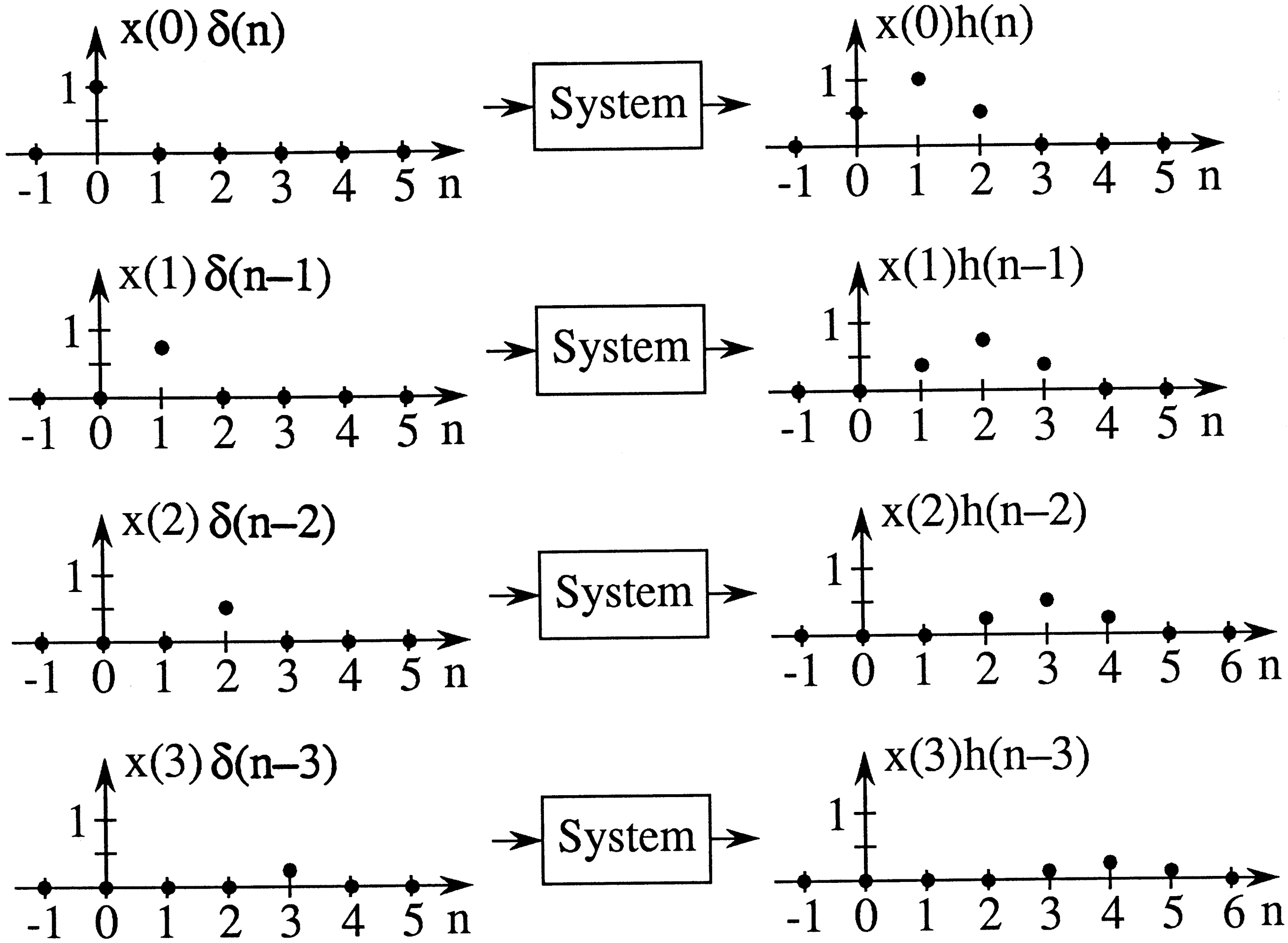
3. Choose impulse as basis function.

Denote *impulse response* by h(n)



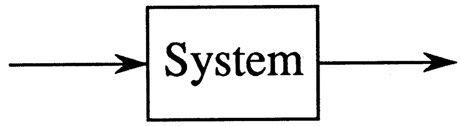
Now consider an arbitrary input x(n).





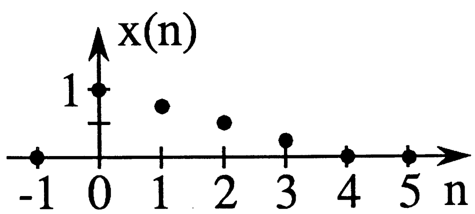
x(0) δ(n)
1
-1 0 1 2 3 4 5 n
System
x(0)h(n)
1
-1 0 1 2 3 4 5 n
x(1) δ(n–1)
1
-1 0 1 2 3 4 5 n
System
x(1)h(n–1)
1
-1 0 1 2 3 4 5 n
x(2) δ(n–2)
1
-1 0 1 2 3 4 5 n
System
x(2)h(n–2)
1
-1 0 1 2 3 4 5 6 n
x(3) δ(n–3)
1
-1 0 1 2 3 4 5 n
System
x(3)h(n–3)
1
-1 0 1 2 3 4 5 6 n

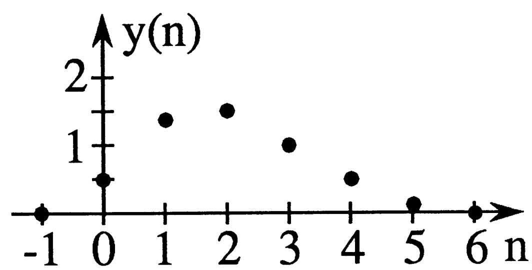
Sum over both the set of inputs and the set of outputs.



$$x(n) = x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3)$$

$$y(n) = x(0)\,h(n) + x(1)\,h(n-1) + x(2)\,h(n-2) + x(3)\,h(n-3)$$





Convolution Sum

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

let $\ell = n - k \Rightarrow k = n - \ell$

$$y(n) = \sum_{\ell=\infty}^{-\infty} x(n-\ell)\,h(\ell) = \sum_{\ell=-\infty}^{\infty} x(n-\ell)\,h(\ell)$$

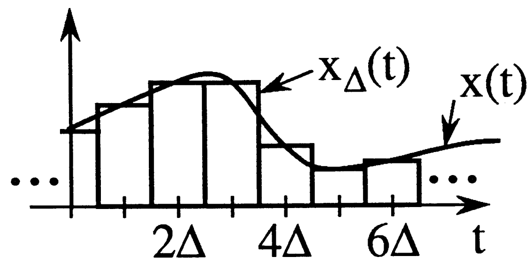
Notation and Identity

For any signals $x_1(n)$ and $x_2(n)$, we use an asterisk to denote their convolution; and we have the following identity

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k)$$
$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\ .$$

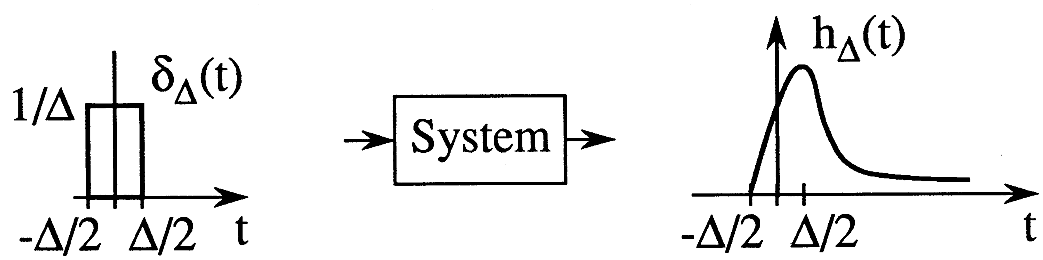
Characterization of CT LTI Systems

1. Approximate x(t) by a superposition of rectangular pulses:



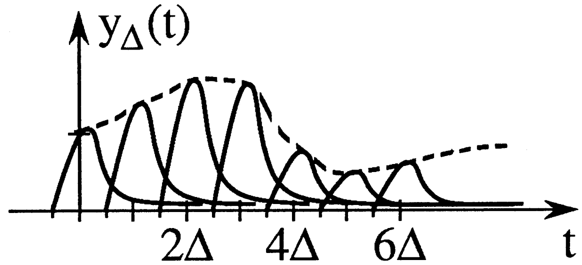
$$x_\Delta(t) = \sum_k x(k\Delta)\, \delta_\Delta(t-k\Delta)\, \Delta$$

2. Find response to a single pulse:



3. Determine response to $x_\Delta(t)$:

$$y_\Delta(t) = \sum_k x(k\Delta)\, h_\Delta(t - k\Delta)\, \Delta$$



Let Δ	$\rightarrow$	$0\ (d\tau)$	$k\Delta$	$\rightarrow$	τ
$\delta_\Delta(t)$	$\rightarrow$	$\delta\ (t)$	$x_\Delta(t)$	$\rightarrow$	$x(t)$
$h_\Delta(t)$	$\rightarrow$	$h\ (t)$	$y_\Delta(t)$	$\rightarrow$	$y(t)$

Convolution Integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$$

Notation and Identity

For any signals $x_1(t)$ and $x_2(t)$, we use an asterisk to denote their convolution; and we have the following identity

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} x_1(t-\tau) x_2(\tau) d\tau \, .$$

Example:

DT System $y(n) = \frac{1}{W} \sum_{k=0}^{W-1} x(n-k)$ W - integer

Find response to $x(n) = e^{-n/D} u(n)$

W – width of averaging window

D – duration of input

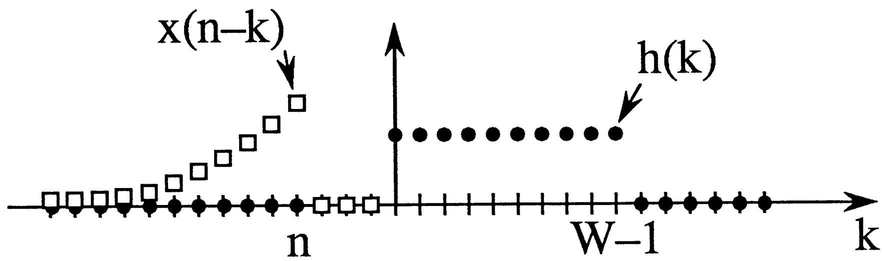
To find impulse response, let $x(n) = \delta(n) \Rightarrow h(n) = y(n)$

$$h(n) = \frac{1}{W} \sum_{k=0}^{W-1} \delta(n-k) = \begin{cases} 1/W, & 0 \le n \le W-1 \\ 0, & \text{else} \end{cases}$$

Now use convolution to find response to $x(n) = e^{-n/D} u(n)$.

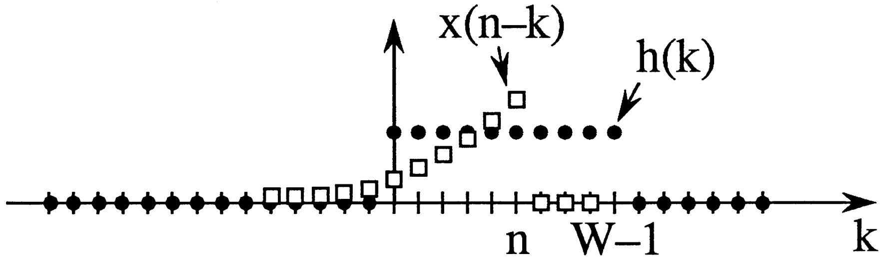
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

Case 1: $n < 0$



$y(n) = 0$

Case 2: $0 \le n \le W-1$



$$y(n) = \sum_{k=0}^{n} x(n-k)\,h(k)$$

$$= \frac{1}{W} \sum_{k=0}^{n} e^{-(n-k)/D}$$

$$= \frac{1}{W}\, e^{-n/D} \sum_{k=0}^{n} e^{k/D}$$

Geometric Series

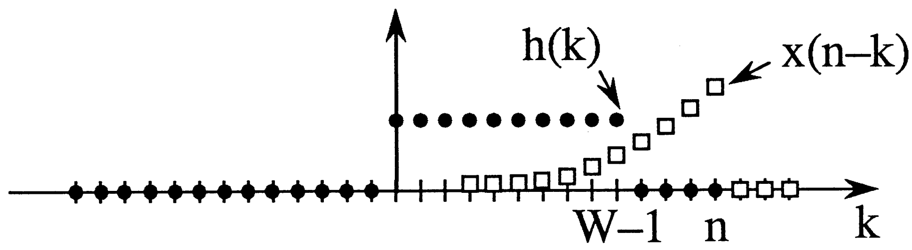
$$\sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z}, \quad \text{for any complex number } z$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \quad |z| < 1$$

$$y(n) = \frac{1}{W} e^{-n/D} \left[\frac{1 - e^{(n+1)/D}}{1 - e^{1/D}} \right]$$

$$= \frac{1}{W} \left[\frac{1 - e^{-(n+1)/D}}{1 - e^{-1/D}} \right]$$

Case 3: $W \leq n$.



$$y(n) = \sum_{k=0}^{W-1} x(n-k)\,h(k)$$

$$= \frac{1}{W} \sum_{k=0}^{W-1} e^{-(n-k)/D}$$

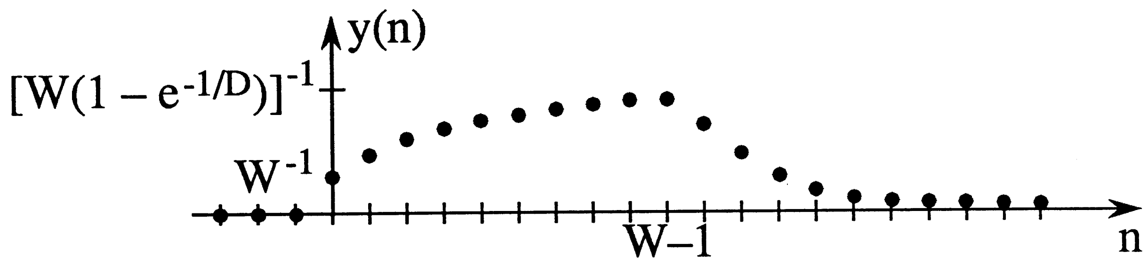
$$= \frac{1}{W}\, e^{-n/D} \sum_{k=0}^{W-1} e^{k/D}$$

$$y(n) = \frac{1}{W} e^{-n/D} \left[\frac{1 - e^{W/D}}{1 - e^{1/D}} \right]$$

$$= \frac{1}{W} \left[\frac{1 - e^{-W/D}}{1 - e^{-1/D}} \right] e^{-[n-(W-1)]/D}$$

Putting everything together

$$y(n) = \begin{cases} 0, & n < 0 \\ \frac{1}{W}\left[\frac{1 - e^{-(n+1)/D}}{1 - e^{-1/D}}\right], & 0 \leq n \leq W-1 \\ \frac{1}{W}\left[\frac{1 - e^{-W/D}}{1 - e^{-1/D}}\right] e^{-[n-(W-1)]/D}, & W \leq n \end{cases}$$



Causality for LTI Systems

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n-k) + \sum_{k=n+1}^{\infty} x(k)\,h(n-k)$$

contribution from past and present inputs	contribution from future inputs

System will be causal $\Leftrightarrow$ second sum is zero for any input $x(k)$.

This will be true $\Leftrightarrow h(n-k) = 0, \quad k = n+1, \ldots, \infty$

$$\Leftrightarrow h(k) = 0, \quad k < 0$$

$\therefore$ A LTI system is *causal* $\Leftrightarrow h(k) = 0,\ k < 0$,
i.e. the impulse response is a *causal signal*

Stability for LTI Systems

Suppose the input is bounded, *i.e.* $M_x < \infty$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$|\, y(n) \,| = |\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \,|$$

$$\leq \sum_{k=-\infty} |\, x(k) \,| \;\; |\, h(n-k) \,|$$

$$= M_x \sum_{k=-\infty}^{\infty} |\, h(k) \,|$$

$\therefore$ It is sufficient for BIBO stability that the impulse response be absolutely summable.

Is it necessary?

Suppose $\sum_{k} |h(k)| \not< \infty$

Consider $y(0) = \sum_{k} x(k)\,h(-k)$

Assuming h(k) is real-valued, let

$$x(k) = \begin{cases} 1, & h(-k) > 0 \\ -1, & h(-k) < 0 \end{cases}$$

then $y(0) = \sum_k |h(k)| \nless \infty$

$\therefore$ A LTI system is *BIBO stable* $\Leftrightarrow$ the impulse response is absolutely summable, *i.e.* $\sum_k |h(k)| < \infty$

Example

$$y(n) = x(n) + y(n-1)$$

Find the impulse response.

Let $x(n) = \delta(n)$, then $h(n) = y(n)$

Need to find solution to

$$y(n) = \delta(n) + 2\,y(n-1)$$

This example differs from earlier ones because the system is recursive, *i.e.* the current output depends on previous output values as well as the current and previous inputs.

1. must specify initial conditions for the system (assume $y(-1) = 0$).

2. cannot directly write a closed form expression for $y(n)$.

Find output sequence term by term

$$
\begin{aligned}
y(0) &= \delta(0) + 2y(-1) &&= 1 + 2(0) &&= 1 \\
y(1) &= \delta(1) + 2y(0) &&= 0 + 2(1) &&= 2 \\
y(2) &= \delta(2) + 2y(1) &&= 0 + 2(2) &&= 4
\end{aligned}
$$

Recognize general form

$$h(n) = y(n) = 2^n u(n)$$

1. Assuming system is initially at rest, it is causal.

2. $\sum_n |h(n)| \nless \infty \Rightarrow$ system is not BIBO stable.