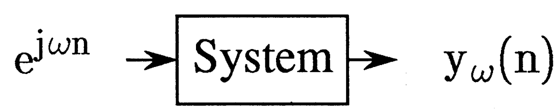
1.2.3 FREQUENCY RESPONSE

Have seen that the impulse signal provides a simple way to characterize the response of an LTI system to *any* input.

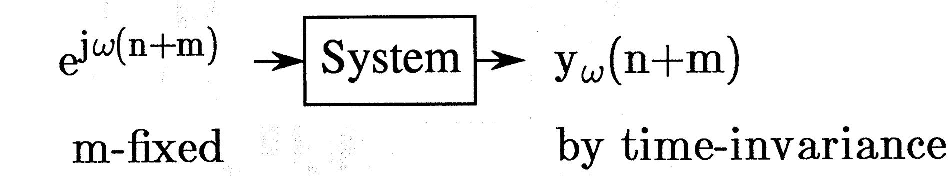
Sinusoids play a similar role.

Consider $x(n) = e^{j\omega n}$, ω - fixed

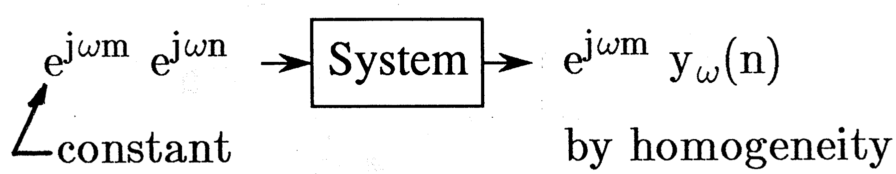
Denote response by $y_\omega(n)$



Now consider



But also



Since $e^{j\omega m}\ e^{j\omega n} = e^{j\omega(n+m)}$,

$$y_\omega(n+m) = y_\omega(n)\ e^{j\omega m}$$

Let n = 0,

$$y_\omega(m) = y_\omega(0)\ e^{j\omega m} \quad \text{for all m}$$

$\therefore$ For a system which is homogeneous and time-invariant, the response to a complex exponential input $x(n) = e^{j\omega n}$ is $y(n) = y_\omega(0)x(n)$, a frequency-dependent constant times the input x(n).

Thus complex exponential signals are eigenfunctions of homogeneous, time-invariant systems.

Comments

1. We refer to the constant of proportionality as the *frequency response* of the system and denote it by

$$H(e^{j\omega}) = y_\omega(0)$$

2. We write it as a function of $e^{j\omega}$ rather than ω for two reasons:

   a. digital frequencies are only unique modulo 2π.

   b. because of the relation between frequency response and the Z transform

3. Note that we did not use superposition. We will need it later when we express the response to arbitrary signals in terms of the frequency response.

Magnitude and Phase of Frequency Response

In general, $H(e^{j\omega})$ is complex-valued,

i.e. $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$

Thus for $x(n) = e^{j\omega n}$

$$y(n) = A(\omega)\ e^{j[\omega n + \underline{/\theta(\omega)}]}$$

Suppose response to any real-valued input x(n) is real-valued.

Let $x(n) = \cos(\omega n) = \frac{1}{2}\left[e^{j\omega n} + e^{-j\omega n}\right]$

Assuming superposition holds,

$$y(n) = \frac{1}{2}\left\{H(e^{j\omega})\, e^{j\omega n} + H(e^{-j\omega})e^{-j\omega n}\right\}$$

$$[y(n)]^* = \frac{1}{2}\left\{\left[H(e^{j\omega})\right]^* e^{-j\omega n} + \left[H(e^{-j\omega})\right]^* e^{j\omega n}\right\}$$

$$y(n) = [y(n)]^* \Leftrightarrow H(e^{-j\omega}) = [H(e^{j\omega})]^*$$

Expressed in polar coordinates

$$A(-\omega)\, e^{j\underline{/\theta(-\omega)}} = A(\omega)\, e^{-j\,\underline{/\theta(\omega)}}$$

$$\therefore \quad A(-\omega) = A(\omega) \quad \text{even}$$

$$\theta(-\omega) = -\theta(\omega) \quad \text{odd}$$

Examples

1. $y(n) = \frac{1}{2}\,[x(n) + x(n-1)]$

Let $x(n) = e^{j\omega n}$

$$y(n) = \frac{1}{2}\left[e^{j\omega n} + e^{j\omega(n-1)}\right]$$
$$= \frac{1}{2}\left[1 + e^{-j\omega}\right] e^{j\omega n}$$

$$\therefore \quad H(e^{j\omega}) = \frac{1}{2}\,[1 + e^{-j\omega}]$$

Factor out the half-angle:

$$H(e^{j\omega}) = \frac{1}{2}\, e^{-j\omega/2} \left[e^{j\omega/2} + e^{-j\omega/2} \right]$$

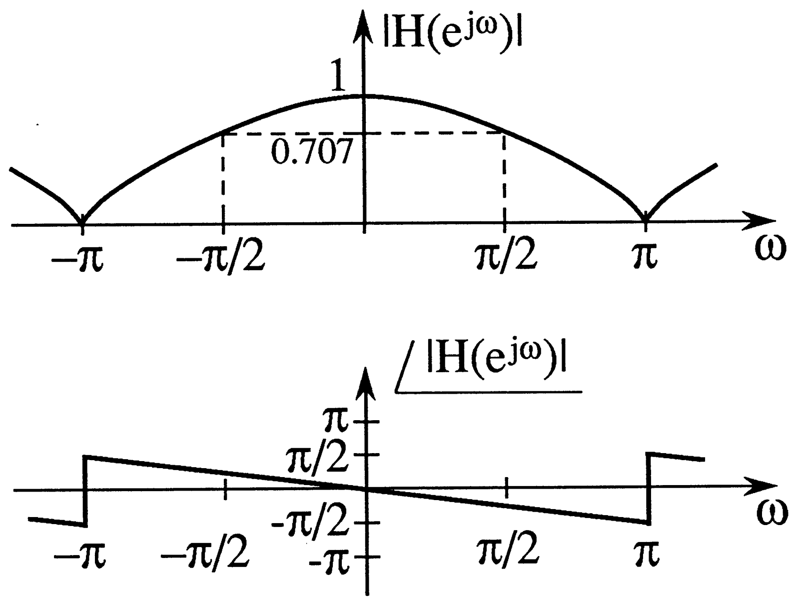
$$= e^{-j\omega/2} \cos(\omega/2)$$

$$|\,H(e^{j\omega})\,| = |\,e^{-j\omega/2}\,| \;\; |\,\cos(\omega/2)\,|$$

$$= |\,\cos(\omega/2)\,|$$

$$\underline{/H(e^{j\omega})} = \underline{/e^{-j\omega/2}} + \underline{/\cos(\omega/2)}$$

$$= \begin{cases} -\,\omega/2, & \cos(\omega/2) \geq 0 \\ -\,\omega/2 \pm \pi, & \cos(\omega/2) < 0 \end{cases}$$



Note: even symmetry of $|H(e^{j\omega})|$

odd symmetry of $\underline{/H(e^{j\omega})}$

periodicity of $H(e^{j\omega})$ with period 2π

low pass characteristic

2. $y(n) = \frac{1}{2}\,[x(n) - x(n-2)]$

Let $x(n) = e^{j\omega n}$

$$y(n) = \frac{1}{2}\left[e^{j\omega n} - e^{j\omega(n-2)}\right]$$

$$= \frac{1}{2}\left[1 - e^{-j\omega 2}\right] e^{j\omega n}$$

$$\therefore \quad H(e^{j\omega}) = \frac{1}{2}\left[1 - e^{-j\omega 2}\right]$$

$$= je^{-j\omega}\left[\frac{1}{j2}\left(e^{j\omega} - e^{-j\omega}\right)\right]$$

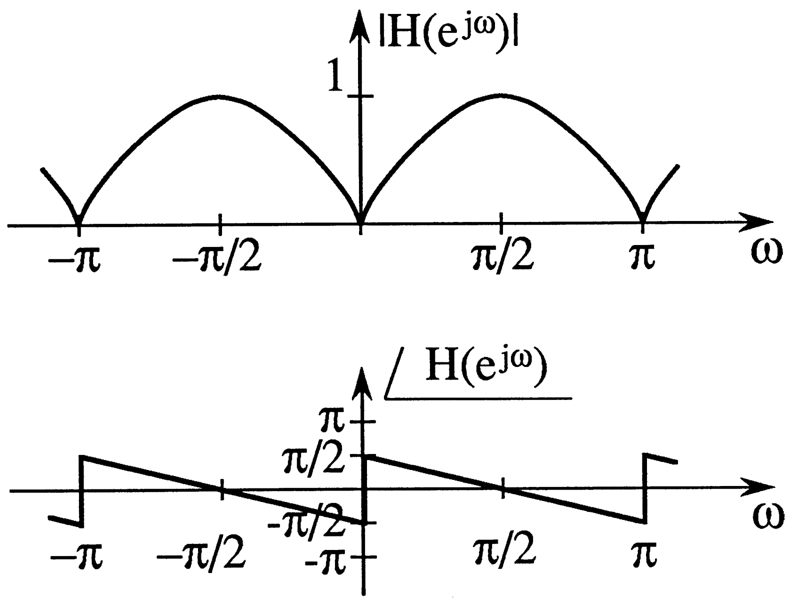
$$H(e^{j\omega}) = je^{-j\omega}\sin(\omega)$$

$$|H(e^{j\omega})| = |j|\ |e^{-j\omega}|\ |\sin(\omega)|$$

$$= |\sin(\omega)|$$

$$\underline{/H(e^{j\omega})} = \underline{/j} + \underline{/e^{-j\omega}} + \underline{/\sin(\omega)}$$

$$= \begin{cases} \pi/2 - \omega, & \sin(\omega) \geq 0 \\ \pi/2 - \omega \pm \pi, & \sin(\omega) < 0 \end{cases}$$



|H(e^{jω})|
1
−π
−π/2
π/2
π
ω
H(e^{jω})
π
π/2
-π/2
-π
−π
−π/2
π/2
π
ω

This filter has a ***bandpass*** characteristic.

Consider $x(n) = \cos(\omega n)$

a. $\omega = 0$

x(n)	...	1	1	1	1	1	...
y(n)	...	0	0	0	0	0	...

↑ n=0

b. $\omega = \pi/2$

x(n)	...	1	0	-1	0	1	0	-1	0	...
y(n)	...	1	0	-1	0	1	0	-1	0	...

n=0

c. $\omega = \pi$

x(n)	...	1	-1	1	-1	1	-1	...
y(n)	...	0	0	0	0	0	0	...

n=0

3. $y(n) = x(n) - x(n-1) - y(n-1)$

Let $x(n) = e^{j\omega n}$, how do we find $y(n)$?

Assume desired form of output,

i.e. $y(n) = H(e^{j\omega})\, e^{j\omega n}$

$$H(e^{j\omega})\, e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-1)} - H(e^{j\omega})\, e^{j\omega(n-1)}$$

$$H(e^{j\omega})\, [1 + e^{-j\omega}]\, e^{j\omega n} = [1 - e^{-j\omega}]\, e^{j\omega n}$$

$$H(e^{j\omega}) = \frac{[1 - e^{-j\omega}]}{[1 + e^{-j\omega}]}$$

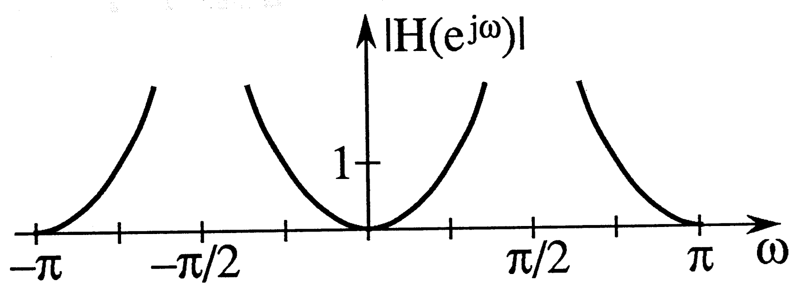
$$= \frac{je^{-j\omega/2}\left[\frac{1}{j2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)\right]}{e^{-j\omega/2}\left[\frac{1}{2}\left(e^{j\omega/2} + e^{j\omega/2}\right)\right]}$$

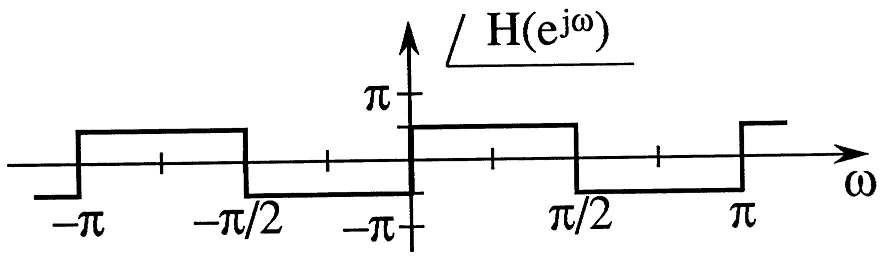
$$= j\,\frac{\sin(\omega/2)}{\cos(\omega/2)}$$

$$= j\,\tan(\omega/2)$$

$$|H(e^{j\omega})| = |\tan(\omega/2)|$$

$$\underline{/H(e^{j\omega})} = \begin{cases} \pi/2, & \tan(\omega/2) \geq 0 \\ \pi/2 \pm \pi, & \tan(\omega/2) < 0 \end{cases}$$





Comments

1. What happens at $\omega = \pi/2$?

2. Factoring out the half angle is possible only for a relatively restricted class of filters.