1.5.2 CONVERGENCE OF THE ZT

- A series

$$\sum_{n=0}^{\infty} u_n$$

is said to *converge* to U if given any real $\in > 0$, there exists an integer M such that

$$\left| \sum_{n=0}^{N-1} u_n - U \right| < \in \quad \text{for all} \quad N > M \, .$$

- Here the sequence u_n and the limit U may be either real or complex.

- A *sufficient* condition for convergence of the series $\sum_{n=0}^{\infty} u_n$ is that it be *absolutely convergent,* *i.e.*

$$\sum_{n=0}^{\infty} |u_n| < \infty .$$

- Note that absolute convergence is not *necessary* for ordinary convergence as illustrated by the following example.

  The series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n} = -\ell n(2)$$

  is convergent, whereas the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not.

- However, when we discuss convergence of the Z Transform, we consider only *absolute* convergence.

  Thus, we say that

$$X(z) = \sum_{n} x(n)\, z^{-n}$$

  converges at $z = z_0$ if

$$\sum_{n} |\, x(n)\, z_0^{-n} \,| = \sum_{n} |\, x(n) \,|\; r_0^{-n} < \infty$$

  where $r_0 = |\, z_0 \,|$

Regions of Convergence for ZT

As a consequence of restricting ourselves to absolute convergence, we have the following properties:

P1. If X(z) converges at $z = z_0$, then it converges for all z for which $|z| = r_0$, where $r_0 = |z_0|$.

proof:

$$\sum_n |x(n)\, z^{-n}| = \sum_n |x(n)|\, r_0^{-n}$$

$$< \infty \quad \text{by hypothesis}$$

P2. If x(n) is a causal sequence, *i.e.* $x(n) = 0$, $n < 0$, and X(z) converges for $|z| = r_1$, then it converges for all z such that $|z| = r > r_1$.

proof:

$$\sum_{n=0}^{\infty} |x(n)\, z^{-n}| = \sum_{n=0}^{\infty} |x(n)|\, r^{-n}$$

$$< \sum_{n=0}^{\infty} |x(n)|\, r_1^{-n}$$

$$< \infty \quad \text{by hypothesis}$$

P3. If $x(n)$ is an anticausal sequence, *i.e.* $x(n) = 0, \quad n > 0$ and $X(z)$ converges for $|z| = r_2$, then it converges for all z such that $|z| = r < r_2$.

proof:

$$\sum_{n=-\infty}^{0} |x(n) z^{-n}| = \sum_{n=0}^{\infty} |x(-n)| r^n$$

$$< \sum_{n=0}^{\infty} |x(-n)| r_2^n$$

$$< \infty \quad \text{by hypothesis}$$

P4. If $x(n)$ is a mixed causal sequence, *i.e.* $x(n) \neq 0$ for some $n < 0$ and $x(n) \neq 0$ for some $n > 0$, and $X(z)$ converges for some $|z| = r_0$, then there exist two positive reals r_1 and r_2 with $r_1 < r_0 < r_2$ such that $X(z)$ converges for all z satisfying $r_1 < |z| < r_2$.

Proof:

Let $x(n) = x_-(n) + x_+(n)$ where $x_-(n)$ is anticausal and $x_+(n)$ is causal. Since $X(z)$ converges for $|z| = r_0$, $X_-(z)$ and $X_+(z)$ must also both converge for $|z| = r_0$.

From Properties 2 and 3, there exist two positive reals r_1 and r_2 with $r_1 < r_0 < r_2$ such that $X_-(z)$ converges for $|z| < r_2$ and $X_+(z)$ converges for $|z| > r_1$.

The ROC for $X(z)$ is just the intersection of these two ROC's.

Example 3

$$x(n) = 2^{-|n|}\ u(n)$$

$$X(z) = \sum_{n} 2^{-|n|}\ z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 2^{n}\ z^{-n} + \sum_{n=0}^{\infty} 2^{-n}\ z^{-n}$$

$$= \sum_{n=0}^{\infty} (2^{-1}\ z)^{n} - 1 + \sum_{n=0}^{\infty} (2^{-1}\ z^{-1})^{n}$$

$$= \frac{1}{1 - 2^{-1}z} - 1 \quad + \quad \frac{1}{1 - 2^{-1}z^{-1}}$$

$$|2^{-1}z| < \infty \qquad\qquad |2^{-1}z^{-1}| < \infty$$

$$\text{or} \quad |z| < 2 \qquad\qquad |z| > 1/2$$

Combining everything

$$X(z) = \frac{-3z^{-1}/2}{1 - 5z^{-1}/2 + z^{-2}} \quad , \quad 1/2 < |z| < 2$$