1.5.6 GENERAL FORM OF RESPONSE OF LTI SYSTEMS

If both X(z) and H(z) are rational

$$Y(z) = H(z)\,X(z)$$

$$= \left[\frac{P_H(z)}{Q_H(z)}\right]\left[\frac{P_X(z)}{Q_X(z)}\right]$$

$$= \left[\frac{P_H(z)}{\prod_{\ell=1}^{N_H}(1 - p_\ell^H z^{-1})}\right]\left[\frac{P_X(z)}{\prod_{\ell=1}^{N_X}(1 - p_\ell^X z^{-1})}\right]$$

$$Y(z) = \frac{P_Y(z)}{\prod_{\ell=1}^{N_Y} (1 - p_\ell^Y z^{-1})}$$

$$P_Y(z) = P_H(z)\, P_X(z)$$

$$N_Y = N_H + N_X$$

p_ℓ^Y is combined set of poles p_ℓ^H and p_ℓ^X

drop superscript/subscript Y

Accounting for poles with multiplicity > 1

$$Y(z) = \frac{P(z)}{\prod_{\ell=1}^{D} (1 - p_\ell z^{-1})^{m_\ell}}$$

D - number of distinct poles

$$N = \sum_{\ell=1}^{D} m_\ell$$

For M < N

$$Y(z) = \sum_{\ell=1}^{D} \sum_{k=1}^{m_\ell} \frac{A_{\ell k}}{(1 - p_\ell z^{-1})^k}$$

- Each term under summation will give rise to a term in the output y(n).
- It will be causal or anticausal depending on location of pole relative to ROC.

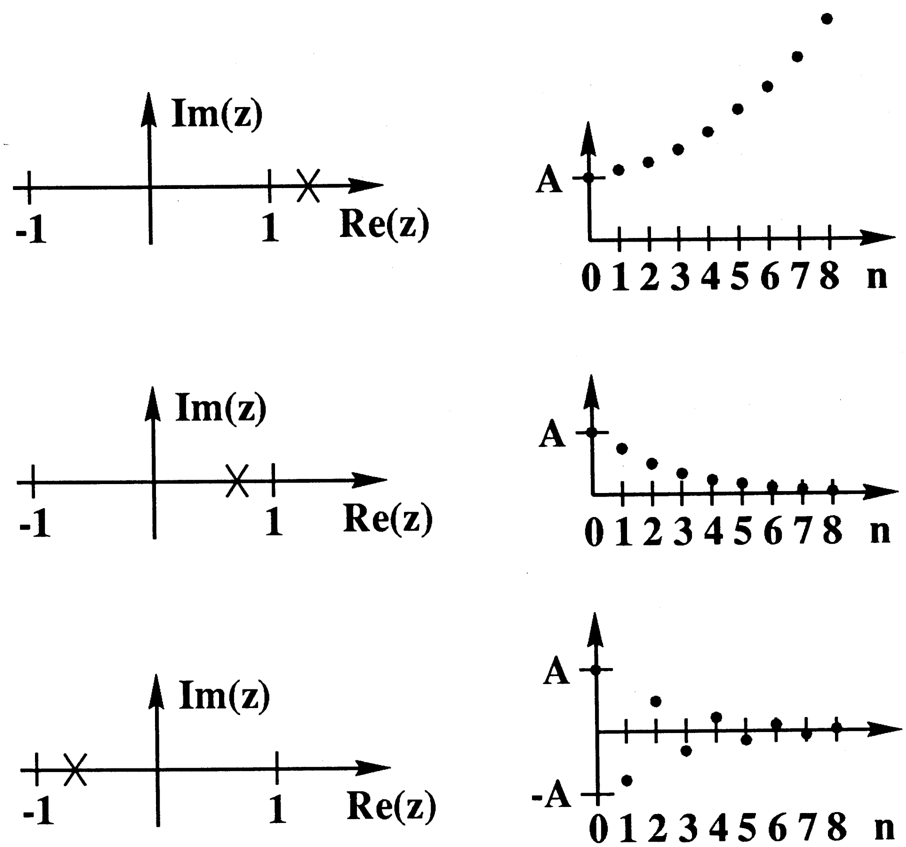
- poles between origin and ROC result in causal terms

- poles separated from origin by ROC result in anticausal terms

- for simplicity, consider only causal terms in what follows

Real pole with multiplicity 1

$$\frac{A}{1 - pz^{-1}} \xrightarrow{ZT^{-1}} Ap^n u(n)$$



Complex conjugate pair of poles with multiplicity 1

$$\frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^*z^{-1}} \xrightarrow{ZT^{-1}} Ap^n u(n) + A^*(p^*)^n\, u(n)$$

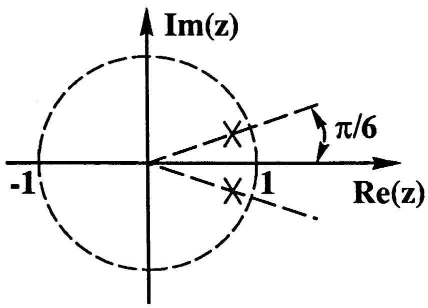
$$[Ap^n + A^*(p^*)^n]\, u(n) =$$

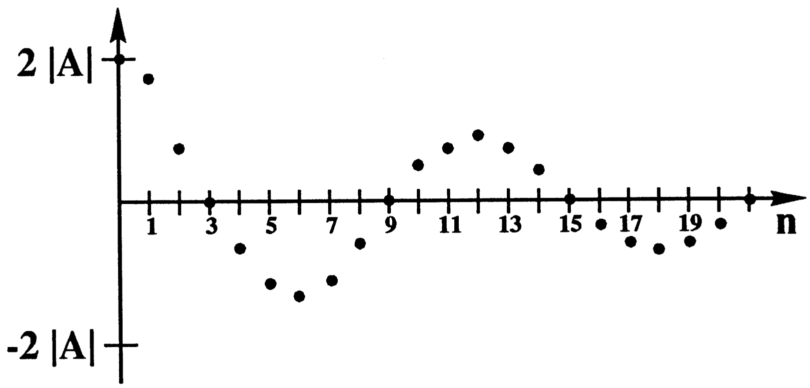
$$[\,|A|\,e^{j\angle A}(\,|p|\,e^{j\angle p})^n + |A|\,e^{-j\angle A}(\,|p|\,e^{-j\angle p})^n]\, u(n)$$

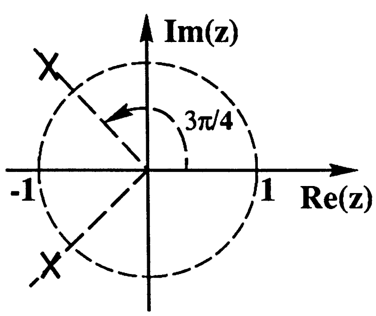
$$= 2\,|A|\;\;|p|^n \cos\,(\angle p n + \angle A)\; u(n)$$

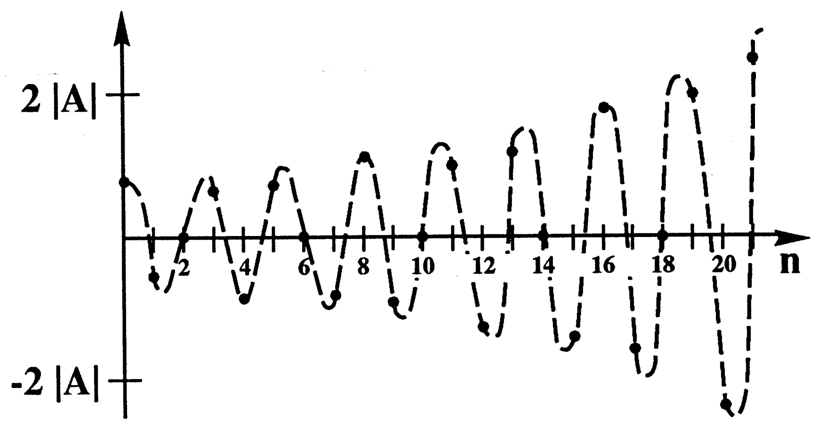
- sinusoid with amplitude $2|A|\,|p|^n$
  - grows exponentially if $|p| > 1$
  - constant if $|p| = 1$
  - decays exponentially if $|p| < 1$
- digital frequency $\omega_d = \angle p$ radians/sample
- phase $\angle A$

Examples


Im(z)
π/6
-1
1
Re(z)


2 |A|
-2 |A|
1
3
5
7
9
11
13
15
17
19
n


Im(z)
3π/4
-1
1
Re(z)


2 |A|
-2 |A|
2
4
6
8
10
12
14
16
18
20
n

Real pole with multiplicity 2

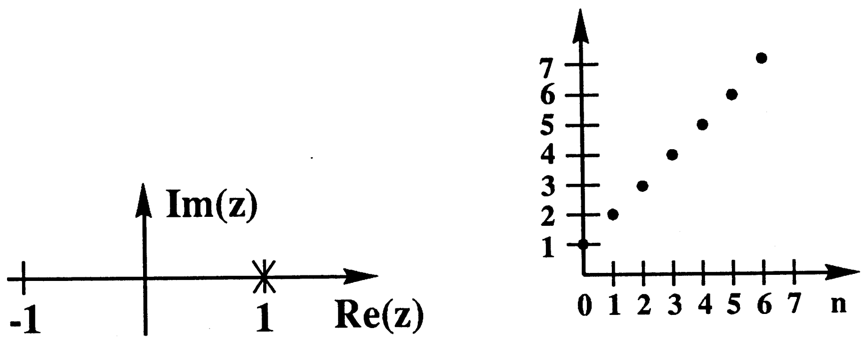
$$\frac{A}{(1 - pz^{-1})^2} \xrightarrow{ZT^{-1}} ?$$

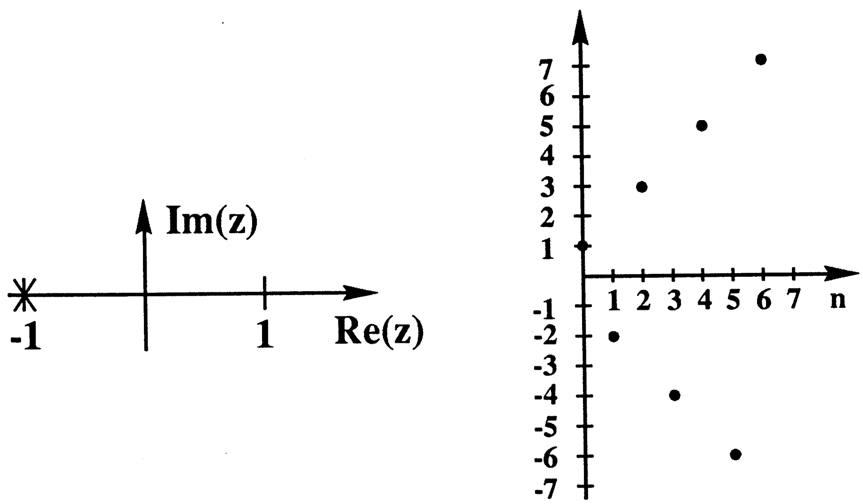
$$\text{recall} \quad na^n u(n) \overset{ZT}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > |a|$$

$$\frac{A}{(1 - pz^{-1})^2} = (A/p)z \left[\frac{pz^{-1}}{(1 - pz^{-1})^2} \right] \rightarrow \frac{A}{p}(n+1)\, p^{n+1} u(n+1)$$

$$\frac{A}{p}(n+1)\ p^{n+1}u(n+1) = A(n+1)\ p^{n}u(n)$$

Examples (A = 1)





- Similar results are obtained with complex conjugate poles that have multiplicity 2.

- In general, repeating a pole with multiplicity m results in multiplication of the signal obtained with multiplicity 1 by a polynomial in n with degree m−1.

Example

$$\frac{A}{(1 - pz^{-1})^3} \xrightarrow{ZT^{-1}} A(n+1)(n+2)\, p^n u(n)$$

Stability Considerations

- A system is BIBO stable if every bounded input produces a bounded output.

- A DT LTI system is BIBO stable $\Leftrightarrow$ $\sum_{n} |h(n)| < \infty$.

- $\sum_{n} |h(n)| < \infty \Leftrightarrow H(z)$ converges on the unit circle.

- A causal DT LTI system is BIBO stable $\Leftrightarrow$ all poles of H(z) are strictly inside the unit circle.

Stability and the general form of the response

1. real pole with multiplicity 1

$$Ap^n u(n) \text{ is bounded if } |p| \leq 1$$

2. complex conjugate pair of poles with multiplicity 1

$$2|A|\,|p|^n \cos(\underline{/p}n + \underline{/A})\, u(n)$$

is bounded if $|p| \leq 1$

3. real pole with multiplicity 2

$$A(n+1)\, p^n u(n) \text{ is bounded if } |p| < 1$$