1.6.1 DERIVATION OF THE DFT

Summary of Spectral Representations

Signal Type	Transform	Frequency Domain	Duality
CT, Periodic	CT Fourier Series (CTFS)	Discrete	No
CT, Aperiodic	CT Fourier Transform (CTFT)	Continuous	Yes
DT, Aperiodic	DT Fourier Transform (DTFT)	Continuous	No
DT, Periodic	DT Fourier Series (DTFS) Discrete Fourier Transform (DFT)	Discrete	Yes

Computation of the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

In order to compute $X(e^{j\omega})$, we must do two things:

1. truncate summation so that it ranges over finite limits,

2. discretize $\omega = \omega_k$

Suppose we choose

1. $\tilde{x}(n) = \begin{cases} x(n)\,, & 0 \leq n \leq N-1 \\ 0\,, & \text{else} \end{cases}$

2. $\omega_k = 2\pi k/N, \quad k = 0,\ldots,N-1$

then we obtain

$$\tilde{X}(k) \triangleq \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi kn/N}$$

This is the forward DFT.

To obtain the inverse DFT, we could discretize the inverse DTFT:

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

$$\omega \rightarrow \omega_k = \frac{2\pi k}{N} \qquad \int_0^{2\pi} d\omega \rightarrow \frac{2\pi}{N} \sum_{k=0}^{N-1}$$

$$X(e^{j\omega}) \rightarrow \tilde{X}(k) \qquad e^{j\omega n} \rightarrow e^{j2\pi k/N}$$

This results in

$$x(n) \approx \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j2\pi kn/N}$$

- Note approximation sign
  - truncated x(n) so $\tilde{X}(k) \approx X(e^{j2\pi k/N})$
  - approximated integral by a sum

Alternate approach

- Use orthogonality of complex exponential signals
- Consider again

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi kn/N}$$

- For fixed $0 \leq m \leq N-1$, multiply both sides by $e^{j2\pi km/N}$ and sum over k.

$$\sum_{k=0}^{N-1} \tilde{X}(k) e^{j2\pi km/N} = \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi kn/N} \right] e^{j2\pi km/N}$$

$$= \sum_{n=0}^{N-1} \tilde{x}(n) \sum_{k=0}^{N-1} e^{-j2\pi k(n-m)/N}$$

$$= \sum_{n=0}^{N-1} \tilde{x}(n) \left[\frac{1 - e^{-j2\pi(n-m)}}{1 - e^{-j2\pi(n-m)/N}} \right]$$

$$\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j2\pi km/N} = \sum_{n=0}^{N-1} \tilde{x}(n)\, N\, \delta(n-m)$$

$$= N\, \tilde{x}(m)$$

$$\therefore \quad \tilde{x}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j2\pi km/N}$$

Summarizing

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

where we have dropped the tildes